Projects and Investigations to accompany Mathematics for Electrical Engineering and Computing

Investigation 1 An investigation of the number 'e'

Introduction

An important equation which describes many physical situations is:-

$$\frac{dy}{dt} = ky \quad \text{(k is a constant)}$$

that is - the derivative of the function is equal to a constant times the function value for all values of t. Examples of models which lead to this differential equation are given in Chapter 8 of Mathematics for Electrical Engineering and Computing. This is an investigation of functions which obey this equation. For the purposes of the investigation we assume no previous knowledge of the number 'e'.

1.1 Use the plotXpose software provided to draw the graph of $y = 2^t$ with $-3 < t < 3$. Print the graph.

1.2 Use the tangent mode in plotXpose to measure the gradients of the tangents to $y=2^t$ at intervals of 0.5. Using a text editor (e.g. Notepad provided with MS Windows) create a table with the following headings:

t	dy/dt	y	(dy/dt)/t

separating each of the columns of values with a space. Save this file as file 'derivative.pts'.

We now want to plot the measured gradients of the tangents against *t*. To do this, delete the headings in the table above. Using your measured values of the gradient of the tangent, plot an approximate graph of $\frac{dy}{dt}$ against *t* by opening 'derivative.pts' in plotXpose.. Only the first two columns of data will be plotted. Print this graph and label it $\frac{dy}{dt}$ against *t* where $y=2^t$

1.3. Examine the ratio $\frac{\frac{dy}{dt}}{t}$ for various *t* values. Is this ratio approximately constant?

1.4. Repeat steps 1 to 3 for $y = 3^t$ (use a *t* range of $-3 < x < 2$)

1.5. If we were to suppose that a number exists, which we could call 'e' , such that the function $y = e^t$ was such that $\frac{dy}{dt} = 1y$ then how could you use your results to justify the assumption that $2 < e < 3$.

1.6. Investigate a graphical, or any other method, of estimating e to one decimal place (assuming that all you know is that $y = e^t$ gives $\frac{dy}{dt} = y$

Investigation 2. Numerical Integration

This is an investigation of numerical integration and the relationship between the error and the step size used.

2.1 Use the software, plotXpose, provided to draw a graph of the function $f(t) = t(t-2)(t+3)$ for $-4 \le t \le 4$

2.2 Perform numerical integration using the Trapezoidal Rule with a starting value of -3 and a step size of 0.1, i.e. calculate an approximation to:

$$A(t) = \int_{-3}^{t} t'(t'-2)(t'+3)\,dt'$$

and plot the approximate graph of A(t) for $-3 \le t \le 4$.

2.3 Record in a table the approximate values of

$$A(t) = \int_{-3}^{t} t'(t'-2)(t'+3)\,dt'$$

for t = -2, -1.5, -1, -0.5, 0, 0.5, 1, 1.5, 2, 2.5 3, 3.5, 4

2.4 Compare your graph of $A(t)$ with the graph of

$$g(t) = \frac{t^4}{4} + \frac{t^3}{3} - 3t^2 + 2$$

In what ways are the graphs of $g(t)$ and $A(t)$ similar? How and why are they dissimilar.

2.5 Find $\int_{-3}^{0} t'(t'-2)(t'+3)\,dt'$ by calculating the exact integral.

2.6 Using the software provided calculate the approximate value of

$$\int_{-3}^{0} t'(t'-2)(t'+3)\,dt'$$

using a step size of h = 1.5, 1, 0.5, 0.3, 0.15, 0.1, 0.05, 0.03, 0.015 and 0.01

2.7 For each value of h used in 2.6 calculate:

(i) The error

(ii) The relative error

2.8 Using the error values obtained in 2.7, draw a graph of error values, ε , against h. Comment on the relationship between ε and h shown in the graph.

2.9 Draw a graph of the relative error, ε_r against h. Using the graph (or otherwise) suggest a step size, h, such that the integral could be assumed to be correct to 6 significant figures. Explain the reasons for your choice and check your result by performing the calculation with your suggested value of h.

2.10 Is it possible to obtain a general formula for the error in terms of the step size h whatever the function *f(t)* we are integrating? What would you expect for Simpson's rule -would the error be larger or smaller than the error in this case?

Investigation 3. Sequences to solve equations

3.1. Using the plotXpose software provided draw a graph of the function y = t^3 - 5t -1

3.2. From the graph estimate the 3 solutions to the equation t^3 - 5t -1 = 0

3.3. Using the software provided and the Newton-Raphson method of solving equations, i.e. the recurrence relation

$$t \leftarrow t - \frac{f(t)}{f'(t)} \quad \text{where } f(t) = t^3 - 5t - 1$$

attempt to find the three solutions of the equation, t^3 - 5t -1 = 0, to 2,4, 6 and 8 significant figures. How many iterations of the algorithm were required in each case? Comment on what happens when the method fails to converge.

3.4. The equation t^3 - 5t -1 = 0 can be rearranged in the following way

$$t^3 - 5t - 1 = 0$$

$$\Leftrightarrow \quad t^3 - 1 = 5t$$

$$\Leftrightarrow \quad t = \frac{t^3 - 1}{5}$$

and the recurrence relation

$$t \leftarrow \frac{t^3 - 1}{5}$$

then provides another method for numerical solution of the equation.

Now use this recurrence relation to attempt to find the three solutions of the equation, t^3 - 5t - 1 = 0, to 2,4, 6 and 8 significant figures. How many iterations of the algorithm were required in each case? Again comment on what happens when the method fails to converge. Compare the rate of convergence of this method and the Newton-Raphson method.

3.4 A fixed point method, $t \leftarrow g(t)$ is known to converge if $|g'(t)| < 1$ when t is near a solution, t^*, such that $t^* = g(t^*)$. Use this to explain those cases where the two methods succeeded or failed to converge.

Investigation 4 An introduction to Systems of Linear Equations

4.1. Using 3 sets of axes draw graphs of the following systems of equations (two equations on each set of axes):

a)

$$\begin{aligned} 2x + 3y &= 2 \\ 6x - y &= 2 \end{aligned}$$

b)

$$\begin{aligned} 9x - 6y &= 12 \\ -6x + 4y &= -8 \end{aligned}$$

c)

$$\begin{aligned} 3x + y &= -1 \\ -6x - 2y &= 3 \end{aligned}$$

4.2. By using the graph list all the solutions of each set of simultaneous equations given in section 4.1.

4.3. Check your graphical solutions by an algebraic solution to each set of equations.

4.4. Can you draw any general conclusions concerning systems of linear equations from the graphical approach? Can you also do this from the algebraic approach?

4.5. Another method of analysing systems of equations is using determinants

A 2x2 determinant is defined in the following manner:

$$\begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad - cb$$

i.e. it takes an arrangement of 4 numbers and results in a single number. Notice that swapping the order of the rows or the columns in the determinant merely changes the sign of the determinant.

In a system of 2 linear equations there are basically 3 sets of determinants, ignoring swapping rows or columns, e.g. system

a)

$$\begin{aligned} 2x + 3y &= 2 \\ 6x - y &= 2 \end{aligned}$$

contains the determinants:

$$\begin{vmatrix} 2 & 3 \\ 6 & -1 \end{vmatrix} \quad \begin{vmatrix} 2 & 2 \\ 6 & 2 \end{vmatrix} \quad \begin{vmatrix} 2 & 3 \\ 2 & -1 \end{vmatrix}$$

4.6. Calculate the three determinants present in each of the three systems of equations given in Section 4.1.

4.7. Using the results of 4.6, investigate a way of analysing systems of equations using determinants.

4.8. Invent 3 other systems of equations which display similar properties to 4.1(a), 4.1(b) and 4.1(c) (but are not equal systems) and check your graphical, algebraic and determinant method of analysing the systems of equations.

4.9. Generalize your results for a general system of equations as below:

$$ax + by = e$$
$$cx + dy = f$$

(where a,b,c,d,e,f are real numbers)

Give a way of deciding whether a system of equations is consistent or inconsistent and whether it is an indeterminate system.

4.10. Investigate a system of three equations and two unknowns e.g.

$$2x + 3y = 1$$
$$4x + y = 2$$
$$3x - 4y = 0$$

Can you solve this system of equations? Are there some such systems that you can/can't solve? Generalise your results if possible.

4.11. Suggest any other interesting lines (or planes!) of investigation.

Project 5 A non-linear Electrical Circuit

A two terminal electrical direct-current device has the voltage-current characteristic shown in Figure 5.1 with the equation $v = f(i)$. A table of values taken from the graph is given in Table 5.1.

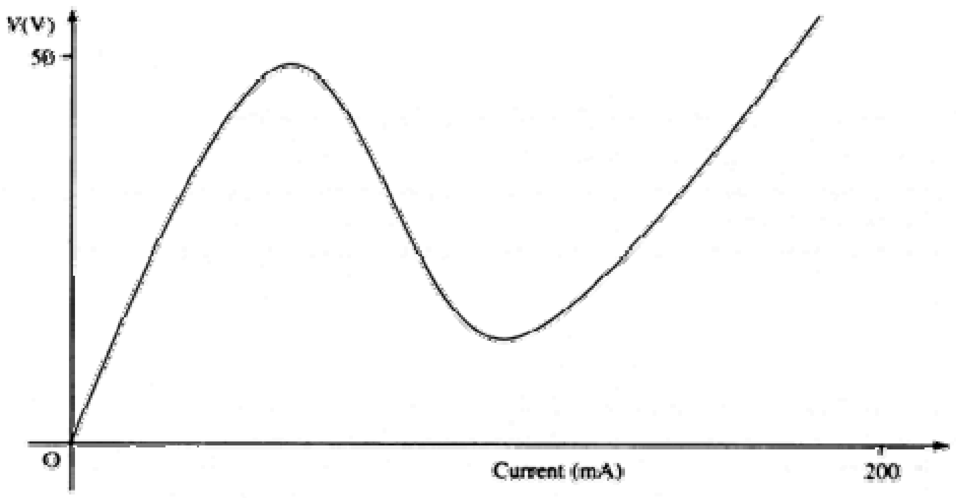

Figure 5.1 The voltage-current characteristic of a non-linear device

Current (mA)	Voltage
0	0
10.417	11.458
20.833	25
31.25	35.938
41.667	44.792
52.083	48.542
54.167	48.958
62.5	46.875
72.92	38.352
83.333	29.167
93.75	18.75
104.16	13.75
108.333	13.547
114.583	14.583
125	18.542
135.417	23.958
145.833	29.688
156.25	35.938
166.667	42.708
177.083	50
187.5	56.25
200	64.583

Table 5.1 Values for the function f(i) read from the graph given in Figure 5.1.

The device is connected in series with a dc voltage source of 50 V and resistor of resistance, 250 Ω. The current *i* in the resulting circuit can be modelled by the equation:

$50 = 250i + f(i)$

5.1. Show by a graphical means or otherwise that the possible steady-state currents are approximately 35, 80 and 125 mA. When an attempt is made to operate the circuit at the equilibrium values it is found that it is only stable at two of the predicted values of the current, 35 and 125 mA. The circuit If an attempt is made to operate at 80 mA the circuit settles down to operate at either 35 or 125 mA.

To improve the model we assume that there is always some residual inductance in an electrical circuit and that for small inductance , L, the *v-i* characteristic still holds. The circuit is now modelled by the equation

$$L\frac{di}{dt} + 250i + f(i) = 50$$

Supposing that i_s is a possible steady-state current; that is $50 = 250i_s + f(i_s)$ and setting $i = i_s + i_p$ where i_p is small. Then a Taylor first-order approximation would give

$$f(i) \approx f(i_s) + i_p f'(i_s)$$

for a region near one of the steady state values. Substitute these assumptions into our equation and use this to explain the unstable behaviour at $i = 80$ mA

5.3. Assuming that at time, $t = 0$, $i \approx 80$ma suggest method(s) to obtain an approximate solution to the differential equation

$$L\frac{di}{dt} + 250i + f(i) = 50$$

using a suitable (small) value for the residual inductance *L*.

Using your method(s) give the approximate time before the circuit has settled to operating at either 35 or 125 mA.

Investigation 6 Probability and the binomial distribution

There is a box of 20 green and 5 red balls. One is picked out of the box at random, recorded and then placed back in the box. If this process is repeated we can represent this using a probability tree.

After performing two trials we have the probability tree as in Figure 6.1.

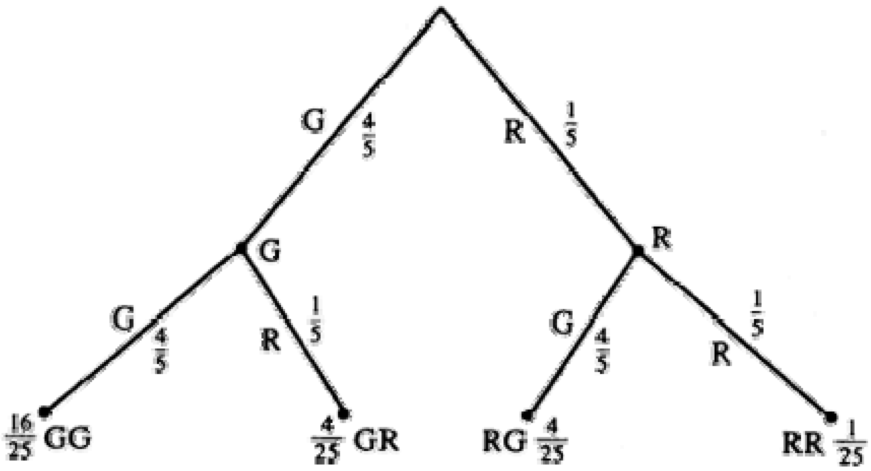

Figure 6.1 Probability tree for 2 trials of picking a ball with replacement from 20 green and five red

This data can also be represented on a table as in Table 6.1.

Outcome	Probability
GG	0.64
GR	0.16
RG	0.16
RR	0.04

Table 6.1 Probability of various outcomes of 2 trials of picking a ball with replacement from 20 green and 5 red

If the order is not considered to be important we can represent the outcomes by the number of greens (the number of 'successes') and hence get Table 6.2.

Outcome	Probability
2	0.64
1	0.32
0	0.04

Table 6.2

Table 6.2 gives us the probability distribution function $P_2(r)$ (the probability of *r* successes in two trials). It can also be represented on a graph as in Figure 6.1

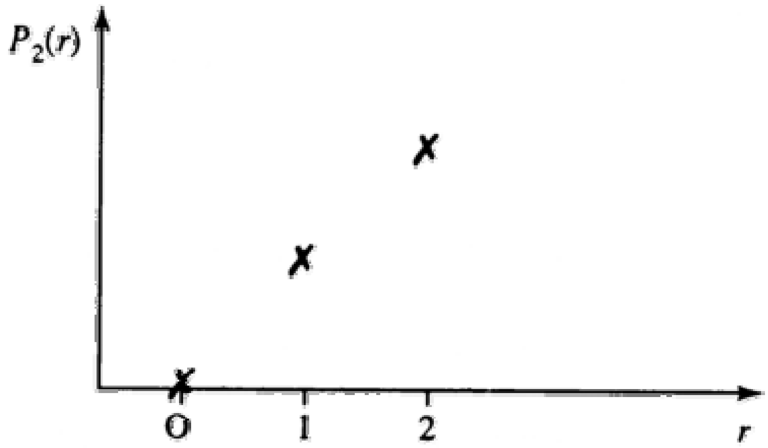

Figure 6.2 The probability distribution $P_2(r)$ for picking a green balls with replacement from 20 green and 5 red

6.1. Draw the probability tree for five trials marking all the probabilities.

6.2. Produce the table of outcomes as in Table 6.1 and in Table 6.2 for five trials.

6.3. Draw the graph of the probability distribution function for five trials.

6.5. Use your results to answer the following questions:

a) What is the probability of selecting four green balls out of five?

b) What is the probability of selecting no green balls out of five?

c) What is the most probable number of green balls selected out of five?

d) What is the probability that two or more balls are red out of the five?

e) Given that the first four balls selected are green what is the probability that the fifth ball selected is red?

6.6. n represents the number of trials and r is the number of green balls after n trials. Given that $\binom{n}{r} = \frac{n!}{(n-r)!r!}$ is the number of combinations of r objects chosen from n, briefly justify (if possible) the following conclusion for our green/red ball problem:

$$P_n(r) = \binom{n}{r}\left(\frac{4}{5}\right)^r\left(\frac{1}{5}\right)^{n-r}$$

6.6. Use the expression given in 6.5 to draw a graph of the probability distribution function for 10 trials

6.7. An electronics company has 10 telephone lines. It is discovered that the average number of lines engaged at any instant is 2.

Calculate the probability that:

a) Exactly 2 lines are engaged.

b) More than 5 lines are engaged.

6.8. There are three assumptions used in the binomial distribution model

a) Each trial is independent.

b) There are two possible outcomes of each trial (success/failure).

c) The probability of success on any one trial is a constant.

Suggest two other examples of the use of the binomial distribution (in particular related to communication or other engineering problems).

6.9. Supposing in our original problem (section 6.1) we had not replaced the ball after each selection:

(a) Draw the probability tree for 3 trials.

(b) Looking at the assumptions of the binomial distribution of Section 6.8 explain why this is not now a binomial distribution?

6.10. Give one communication or other engineering example where the outcome of each trial is success/failure but is not suitable for modelling using the binomial distribution.

Investigation 7. The distribution of the sample mean

Introduction

This is an investigation into the distribution of the sample mean. In order to perform the investigation a lot of data is needs to be collected so it is preferable if organised amongst a group of students. If students work in pairs then sufficient data can be collected in less than 20 minutes. Alternatively a computer simulation could be used.

The result of this investigation illustrates one application of the Central Limit Theorem. That is the sum of a large number of random variables approaches the normal distribution, even when the original random variables are not themselves normal. We also attempt to derive the relationship between the distribution of the sample means and the population mean and standard deviation.

The distribution of the sample mean is important in quality assurance. Supposing a factory claims to produce 30mm nails and accepts a standard deviation of 2mm. Each day a certain sample of the production is measured. The mean of the sample should be 'something near' 30mm but how near is acceptable if the hypothesis that the population is of mean 30mm and standard deviation is 2mm is to be considered reasonable? After this investigation you should be able to answer this question.

The example of a die is used because (a) it is very simple to collect lots of data quickly and (b) the population distribution (of outcomes of throwing a die) is nowhere near normal, and yet the sample means display nearly normal behaviour for quite small numbers in the sample.

7.1 Draw the probability distribution for the outcomes of throwing a fair die. Find the mean and the standard deviation. This is the population distribution.

7.2 Throw a die four (sample size, n = 4) times and find the mean of the four throws. Record the mean and repeat this process about 50 times. So you have 50 data values. Each one found from the mean of four numbers.

7.3 Draw a histogram of the distribution of the sample means found in section 7.2 Find the mean of this data and its standard deviation. (That is the mean, $\overline{x}_n$ and the standard deviation, σ_n of the sample means where the number in the sample is n = 4)

7.4 Repeat 7.2 and 7.3 for sample sizes of 5,6,7,8,9 and 10.

7.5 Comment on the shape of the histograms found in each case.

7.6 Draw a graph of the standard deviation of the sample means, σ_n, against n (the number in the sample) and draw a log-log graph of the same data. Estimate the relationship between σ_n and n.

7.7 Write out your conclusions about the distribution of the sample mean and suggest how this could be used in the quality assurance problem given in the introduction.