



An Introduction to Vectors

Representing Magnitude and Direction

Example

Speed

+

Direction



= Velocity



Scalars vs. Vectors

Some quantities can be described by a single number (magnitude), while others require both a magnitude and a direction.

Scalar Quantities

Represented by a simple number.

- Time
- Distance
- Mass

Vector Quantities

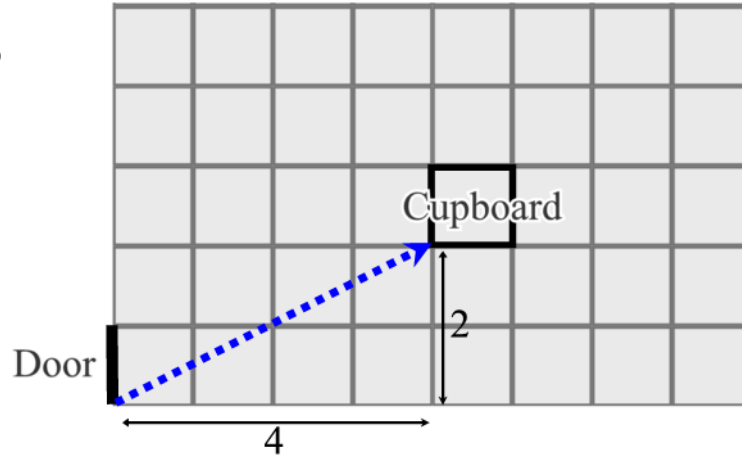
Represented by magnitude and direction.

- Velocity
- Acceleration
- Force
- Displacement

Representing Vectors: A Simple Example

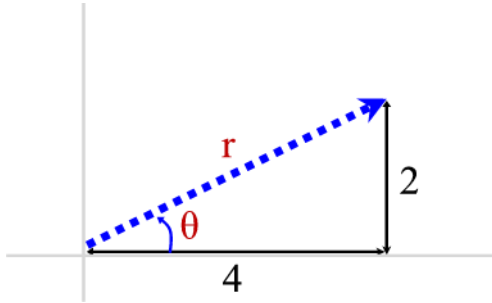
Imagine moving across a tiled room. To get from the door to the cupboard, you might move four tiles to the right and two tiles up. This displacement can be represented by the vector $(4, 2)$.

The order is crucial; moving $(4, 2)$ is different from moving $(2, 4)$.



Magnitude and Direction

For the vector (4, 2), we can calculate its magnitude (length) and define its direction.



Magnitude

The length of the vector, found using Pythagoras' theorem.

$$\text{Magnitude } r = \sqrt{4^2 + 2^2} = \sqrt{20} \approx 4.47$$

Direction

The orientation of the vector, often described by the angle it makes with a reference axis.

$$\theta = \tan^{-1}\left(\frac{2}{4}\right) \approx 0.4636$$

Vector Dimensions and Notation

Vectors can be written as rows or columns. Their dimension is the number of elements they contain.

2D & 3D Vectors

2D vectors like (x, y) represent movement on a flat surface. 3D vectors like (x, y, z) represent movement in space.

Row and Column Notation

A vector can be written as a row $(1, -1, 2)$ or as a column

$$\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

The order of the numbers is always important.

Symbolic Notation

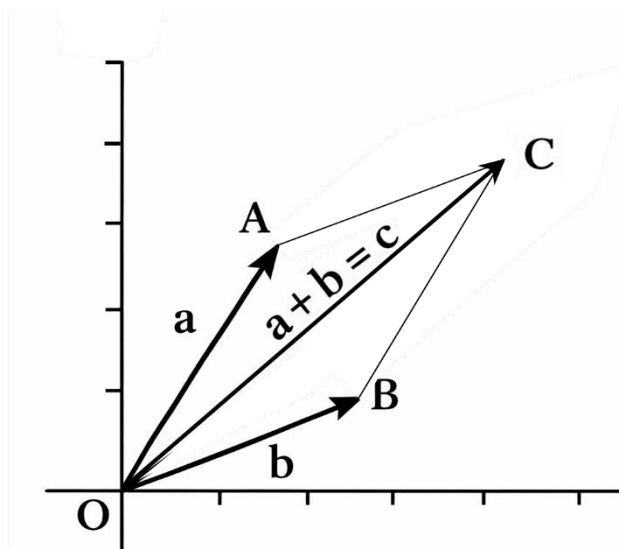
In text, vectors are often represented by bold letters ***a***, underlined letters *a*, or with an arrow above $\overrightarrow{AB}$ where A and B are points at either end of the vector .

Vector Operations: Addition and Subtraction

Addition

To add two vectors, add their corresponding components.

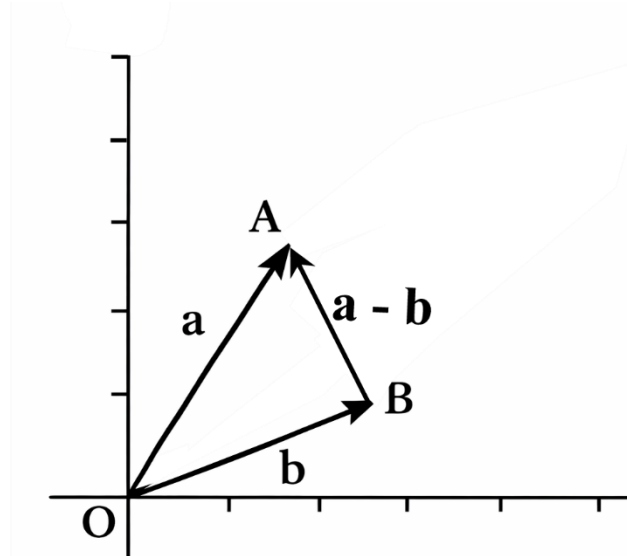
$$(2, 3) + (4, 2) = (2 + 4, 3 + 2) = (6, 5)$$



Subtraction

To subtract a vector, subtract its corresponding components.

$$(2, 3) - (4, 2) = (2 - 4, 3 - 2) = (-2, 1)$$



Polar Coordinates

Instead of (x, y) components, a 2D vector can be described by its length (r) and its angle (θ) relative to the x-axis.

From Rectangular (x, y) to Polar (r, θ)

- Magnitude: $r = \sqrt{x^2 + y^2}$
- Angle: $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ (adjusting for quadrant)

From Polar (r, θ) to Rectangular (x, y)

- x -component: $x = r \cos(\theta)$
- y -component: $y = r \sin(\theta)$

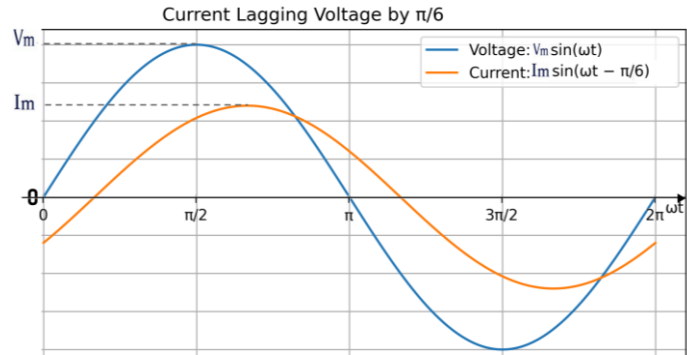
Note: To add vectors in polar form, first convert them to rectangular form, add the components, then convert back.

Application of Vectors to Represent Waves (phasors)

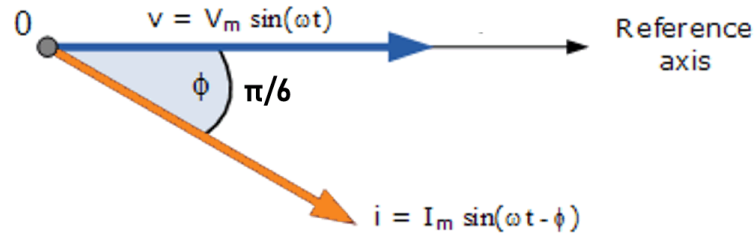
Waves of a fixed frequency can be represented by phasors giving the amplitude and phase.

$f(t) = A \cos(\omega t + \varphi)$ can be represented by its amplitude and phase $A \angle \varphi$.

Electrical Circuits Example



Phasor Diagram of a Sinusoidal Waveform



The current phasor lags the voltage phasor by the angle (Φ)

Converting $f(t) = A \cos(\omega t + \varphi)$ to rectangular form gives (c, d) where $f(t) = c \cos(\omega t) - d \sin(\omega t)$.

Using ideas of conversion from polar to rectangular form and vector addition, waves of the same frequency can be easily combined.

Scalar Multiplication and Unit Vectors

Scalar Multiplication

Multiplying a vector by a scalar changes its length but not its direction. For example, $3\mathbf{a}$ is three times as long as $\mathbf{a}$.

$$3(1, -2) = (3, -6)$$

Unit Vectors

A unit vector has a length of 1 and is used to specify a direction. To find the unit vector ($\hat{\mathbf{r}}$) for a vector ($\mathbf{r}$), divide $\mathbf{r}$ by its own magnitude.

$$\hat{\mathbf{r}} = \frac{\mathbf{r}}{|\mathbf{r}|}$$

The Scalar (Dot) Product

The scalar product (or dot product) of two vectors results in a single number (a scalar).

Calculation

$$\mathbf{a} = (a_1, a_2), \mathbf{b} = (b_1, b_2)$$

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2$$

Example: $(2, 3) \cdot (1, 2) = (2)(1) + (3)(2) = 8$

Geometric Meaning

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos(\theta)$$

This is useful for finding the angle (θ) between two vectors.

If $\mathbf{a} \cdot \mathbf{b} = 0$, the vectors are at right angles (orthogonal).

The Vector (Cross) Product

Definition

The vector product (or cross product) creates a new vector that is perpendicular to the plane containing the two original vectors.

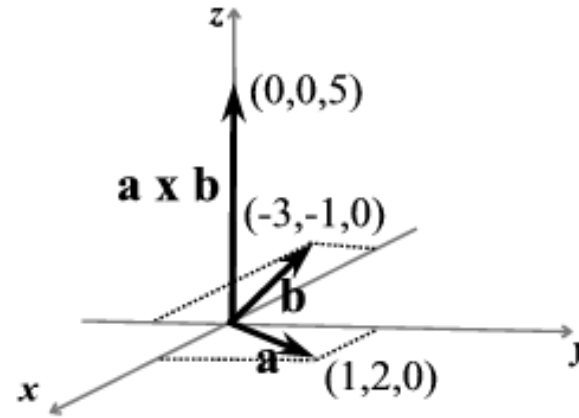
$$(a_1, a_2, 0) \times (b_1, b_2, 0) = (0, 0, a_1b_2 - a_2b_1) = (a_1b_2 - a_2b_1)\mathbf{k}$$

where $\mathbf{k}$ is the unit vector in the z-direction.

Application: Area

The magnitude of the cross product, $|\mathbf{a} \times \mathbf{b}|$, is equal to the area of the parallelogram formed by vectors $\mathbf{a}$ and $\mathbf{b}$.

$$|\mathbf{a} \times \mathbf{b}| = ab \sin(\theta)$$



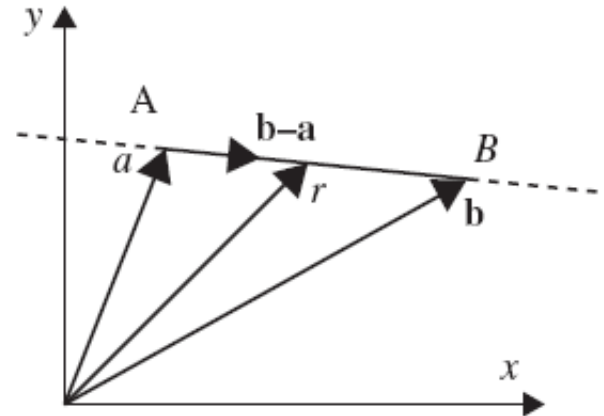
Vector Equation of a Line

The equation of a line can be expressed using vectors. It requires a point on the line and a vector that gives the direction of the line.

The Formula

$$\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a})$$

- $\mathbf{r}$ is any point on the line
- $\mathbf{a}$ is the position vector of a known point on the line
- $\mathbf{b}$ is the position vector of another known point on the line
- $(\mathbf{b} - \mathbf{a})$ is the direction vector of the line
- λ is a scalar parameter



Summary

- Vectors possess both magnitude and direction. Scalars have magnitude only.
- Vector operations include addition, subtraction, and scalar multiplication.
- Unit vectors specify direction with a magnitude of 1.
- The Scalar (Dot) Product results in a number and can be used to find angles between two vectors.
- The Vector (Cross) Product results in a new vector, perpendicular to the plane of the original vectors, and the magnitude of the cross product can be used to find the area of the parallelogram.
- A 2D Vectors can be represented in polar form by finding its magnitude (length) and angle it makes to a reference axis:
 (r, θ) or $r \angle \theta$
- Waves of a fixed frequency can be represented by phasors giving the amplitude and phase. This representation of waves as phasors mean simple vector addition and algebra can be used.
- The vector equation of a line can be found using the position vector of any point on the line and its direction vector.