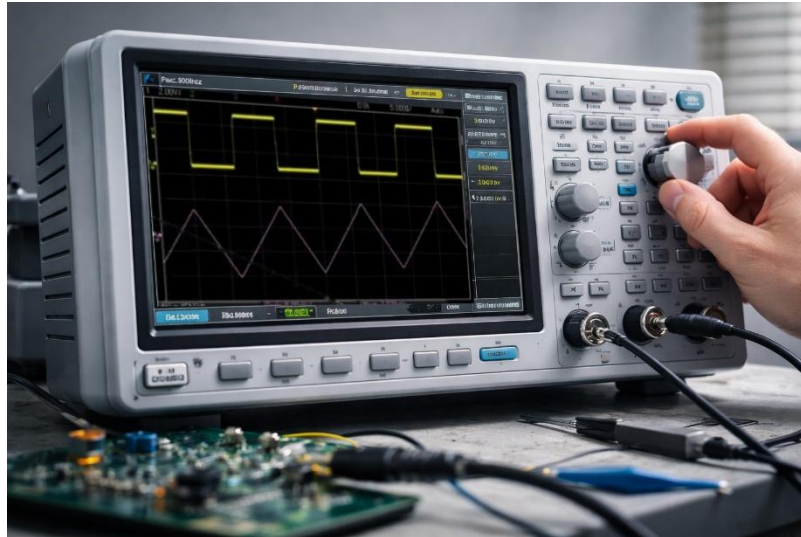




Integration



What is Integration?

Integration is the inverse process of differentiation. If we know the rate of change of a quantity (its derivative), integration allows us to find the original quantity. For example, while velocity is the derivative of distance with respect to time, distance is the integral of velocity with respect to time.

Function	→	Derivative
Integral	←	Function
$x^2 + C$		$2x$
$\sin(x) + C$		$\cos(x)$

These can be written as:

$$\frac{d}{dx}(x^2 + C) = 2x \Leftrightarrow \int 2x \, dx = x^2 + C$$

$$\frac{d}{dx}(\sin(x) + C) = \cos(x) \Leftrightarrow \int \cos(x) \, dx = \sin(x) + C$$

Two Key Challenges in Integration

1. The Constant of Integration

The derivative of any constant is zero. This means functions (like $x^2 + 1$, $x^2 - 2$, $x^2 + 1000$) all have the same derivative ($2x$). When we integrate, we find a family of possible functions that differ by a constant, C . This is called the indefinite integral.

Once the unknown constant is replaced by some value to fit a certain problem; we have the particular integral.

2. Non-Integrable Functions

Many seemingly simple functions, such as $\frac{\sin(x)}{x}$, cannot be integrated exactly using standard formulas. This makes numerical methods of integration particularly important.

Numerical methods depend on understanding the idea of integration as area under the graph. The definite integral of a function, $y = f(x)$, is the integral between two values of x and therefore gives a number (not a function of x) as a result. There is no uncertainty, hence it is called the definite integral.

General vs. Particular Solutions

To find the specific constant 'C' for a particular problem, we use known information or initial conditions.

General Solution

This is the result of integration including '+ C'. For a ball with velocity $v = 20 - 10t$, the general solution for distance is

$$s = 20t - 5t^2 + C.$$

It represents all possible solutions.

Particular Solution

If we know the ball was thrown from the ground ($s = 0$ at $t = 0$), we can solve for C . Substituting gives $0 = 0 - 0 + C$, so $C = 0$.

The particular solution is

$$s = 20t - 5t^2.$$

Finding Integrals: Standard Forms

A table of standard integrals can be created by reversing a table of standard derivatives and adding the constant of integration, C .

$f(x)$	$\int f(x) dx$
$x^n (n \neq -1)$	$\frac{x^{n+1}}{(n+1)} + C$
1	$x + C$
$\cos(x)$	$\sin(x) + C$
$\sin(x)$	$-\cos(x) + C$
$\frac{1}{1+x^2}$	$\tan^{-1}(x) + C$
$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1}(x) + C$
$-\frac{1}{\sqrt{1-x^2}}$	$\cos^{-1}(x) + C$

Integration by Substitution

This method, which reverses the differentiation chain rule, simplifies an integral by substituting part of the expression with a new variable, 'u'.

Linear Form: $\int f(ax + b) dx$

make the substitution $u = ax + b$.

Example Find $\int \sin(3x + 2) dx$.

Solution Substitute $u = 3x + 2$. Then $du/dx = 3 \Rightarrow du = 3 dx \Rightarrow dx = du/3$. Then the integral becomes

$$\int \sin(u) \frac{du}{3} = -\frac{\cos(u)}{3} + C$$

Re-substitute $u = 3x + 2$ to give

$$\int \sin(3x + 2) dx = -\frac{\cos(3x + 2)}{3} + C.$$

Product Form: $\int f(u) (du/dx) dx$

Look for a function multiplied by its derivative.

Example Find $\int x \sin(x^2) dx$.

Solution Substitute $u = x^2 \Rightarrow du/dx = 2x \Rightarrow du = 2x dx \Rightarrow dx = du/2x$ to give

$$\begin{aligned} \int x \sin(x^2) dx &= \int x \sin(u) \frac{du}{2x} = \int \frac{1}{2} \sin(u) du \\ &= -\frac{1}{2} \cos(u) + C. \end{aligned}$$

As $u = x^2$, we have

$$\int x \sin(x^2) dx = -\frac{1}{2} \cos(x^2) + C.$$

Integration by Parts

This integration by parts formula can be justified from the product rule for differentiation. This technique is used to integrate products of functions, like $\int x \sin(x) \, dx$.

The Formula: $\int u \, dv = uv - \int v \, du$

- Split the integrand into two parts: ' u ' and ' dv '.
- Choose ' u ' to be something that simplifies when differentiated (e.g., x).
- Choose ' dv ' to be the part you can integrate (e.g., $\sin(x) \, dx$).
- Find du (by differentiating u) and v (by integrating dv).
- Substitute these parts into the formula and solve the new, simpler integral.

Example Find $\int x \sin x \, dx$

Solution Use $u = x$; $dv = \sin(x) \, dx$. Then

$$\frac{du}{dx} = 1 \quad \text{and} \quad v = \int \sin x \, dx = -\cos(x).$$

Substitute in $\int u \, dv = uv - \int v \, du$ to give

$$\begin{aligned} \int x \sin x \, dx &= -x \cos(x) - \int -\cos(x) 1 \, dx \\ &= -x \cos(x) + \sin(x) + C. \end{aligned}$$

Integration Using Trigonometric Identities

Powers of trigonometric functions can often be integrated by first rewriting them using identities.

Even Powers (e.g., $\sin^2 x$)

Use double angle formulas to reduce the power.

Example 7.13 Find $\int \sin^2(x) \, dx$.

Solution As $\cos(2x) = 1 - 2 \sin^2(x)$,

$$\sin^2(x) = \frac{1 - \cos(2x)}{2}.$$

The integral becomes

$$\int \sin^2(x) \, dx = \int \frac{1 - \cos(2x)}{2} \, dx = \frac{1}{2}x - \frac{1}{4} \sin(2x) + C.$$

Odd Powers (e.g., $\cos^3 x$)

Use the identity

$$\cos^2(x) = 1 - \sin^2(x) \text{ or } \sin^2(x) = 1 - \cos^2(x)$$

Example 7.14 Find $\int \cos^3(x) \, dx$.

Solution As $\cos^2(x) + \sin^2(x) = 1$, we have $\cos^2(x) = 1 - \sin^2(x)$.

$$\begin{aligned} \int \cos^3(x) \, dx &= \int \cos(x) \cos^2(x) \, dx \\ &= \int \cos(x) (1 - \sin^2(x)) \, dx \\ &= \int \cos(x) - \cos(x) \sin^2(x) \, dx \\ &= \int \cos(x) \, dx - \int \cos(x) \sin^2(x) \, dx. \end{aligned}$$

The second part of this integral is of the product form $\int f(u) \frac{du}{dx} \, dx$

so can be solved using substitution.

Applications of Integration

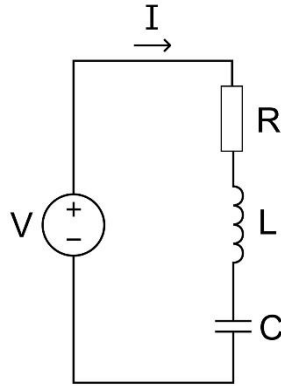


Mechanics

Distance / velocity: $x = \int v \, dt$

Velocity / acceleration: $v = \int a \, dt$

Work done / force: $W = \int F \, dx$

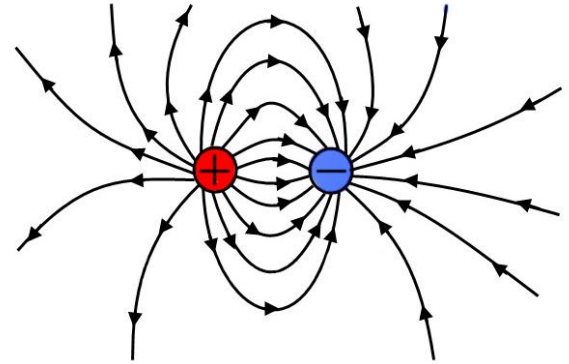


Electrical Circuits

Charge / current: $Q = \int I \, dt$

Current / voltage drop across an inductor:

$$I = (1/L) \int V \, dt$$



Electrostatics & Gases

Electric potential / Electric field:

$$V = -\int E \, dx$$

Work under isothermal expansion of a gas / volume:

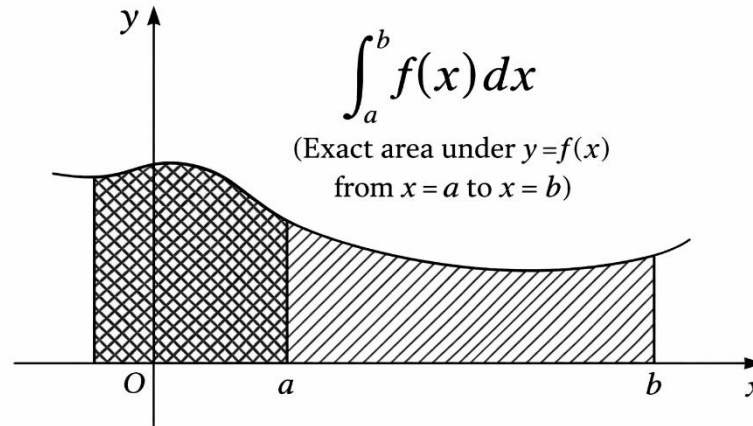
$$W = \int p \, dV$$

The Definite Integral: Area Under a Curve

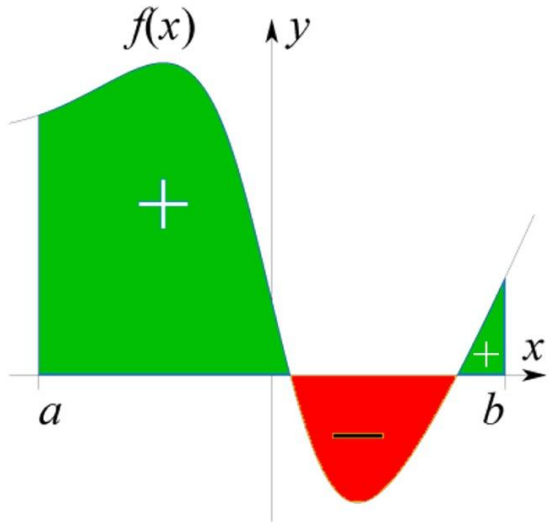
The definite integral, written as $\int_a^b f(x) dx$, represents the exact area under the curve $y = f(x)$ from $x = a$ to $x = b$. It results in a single numerical value.

Calculation

- Find the indefinite integral, $F(x)$
- Evaluate it at the upper limit: $F(b)$
- Evaluate it at the lower limit: $F(a)$
- Subtract: Area = $F(b) - F(a)$
- The constant of integration always cancels out.



Positive and Negative Areas



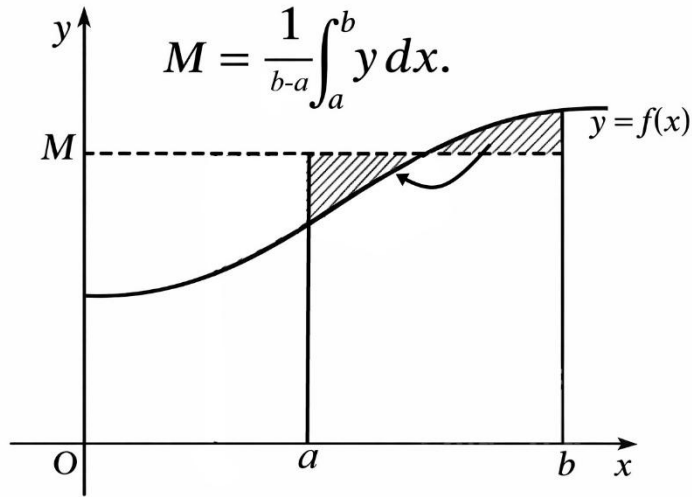
The definite integral calculates the *net* area. Areas below the x-axis are counted as negative.

- If the curve is below the x-axis, the integral will be negative. The physical area is the absolute value of the integral.
- If a curve crosses the x-axis within the integration interval, the positive and negative areas can cancel each other out.
- To find the total physical area, you must split the integral at the points where the curve crosses the x-axis and sum the absolute values of each part.

Mean and R.M.S. Values

Mean Value

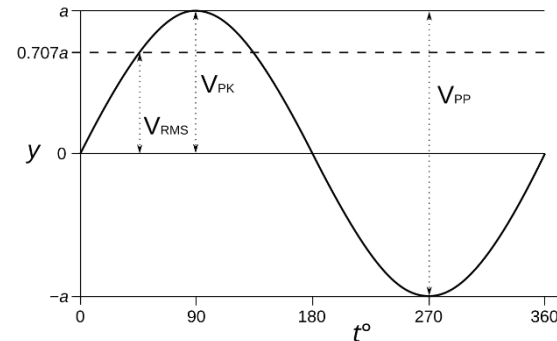
The average height of the function over an interval. It's the value a constant function would need to have the same area



Root Mean Squared (R.M.S.) Value

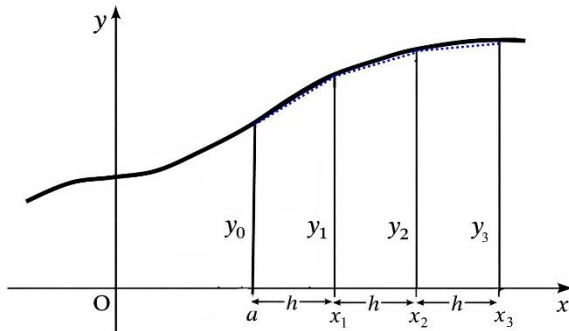
The square root of the mean of the squared function. It's crucial for AC circuits, as it provides a measure of effective voltage or current that relates to power, and is always positive.

$$\text{r.m.s.}(y) = \sqrt{\frac{1}{b-a} \int_a^b y^2 \, dx}$$



Numerical Integration

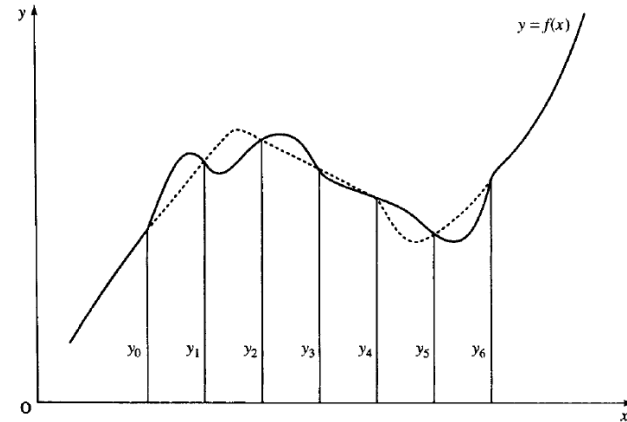
When an integral cannot be solved analytically (by formula), we use numerical methods to approximate the area by dividing it into strips.



Trapezoidal Rule

The area is divided into vertical strips, and each strip is approximated as a trapezoid. The formula is

$$A = h(\frac{1}{2}y_0 + y_1 + \dots + y_{n-1} + \frac{1}{2}y_n)$$



Simpson's Rule

This method approximates the curve over pairs of strips using a parabola. It is generally more accurate for the same number of strips. The formula is

$$A = (h/3)(y_0 + 4y_1 + 2y_2 + \dots + 4y_{n-1} + y_n)$$

Summary

- Integration is the inverse process of differentiation
- The indefinite integral has a constant of integration, '+ C '. A particular solution is found when ' C ' is determined from known conditions.
- Key integration techniques include Substitution, Integration by Parts, and using Trigonometric Identities.
- The definite integral calculates a specific numerical value, $F(b) - F(a)$, representing the area under the curve.
- Mean and R.M.S. values are used to express the average behaviour of a function.
- Trapezoidal and Simpson's rules are two numerical methods used for approximating integrals that cannot be solved analytically.