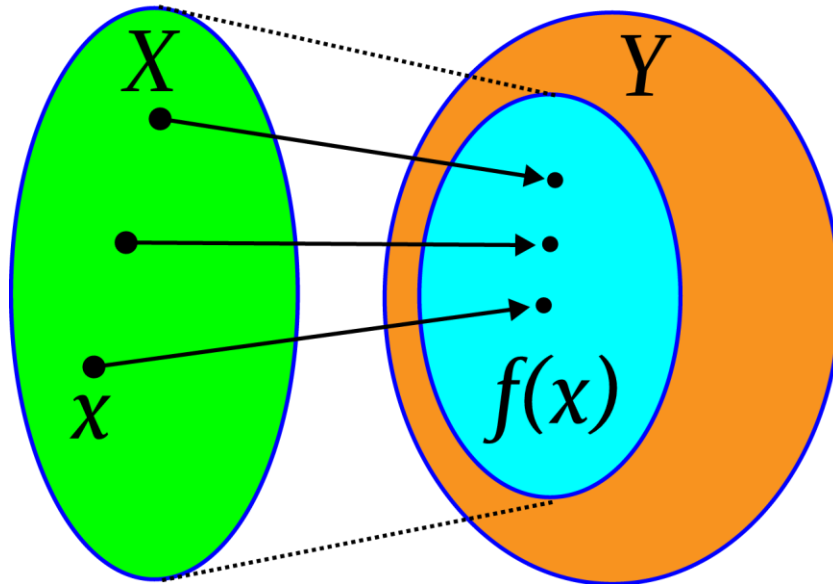
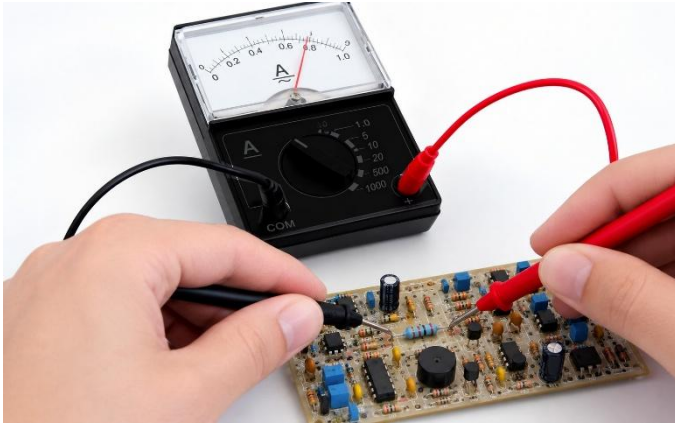


Sets and functions





1.1 Introduction: Functions in Engineering

Finding relationships between quantities is central to engineering. For instance, in a simple circuit with a $1000\ \Omega$ resistor, Ohm's law gives the relationship between current (I) and voltage (V) as $I = V / 1000$. For any value of V , we can find a corresponding value of I . I is a function of V .



Functions

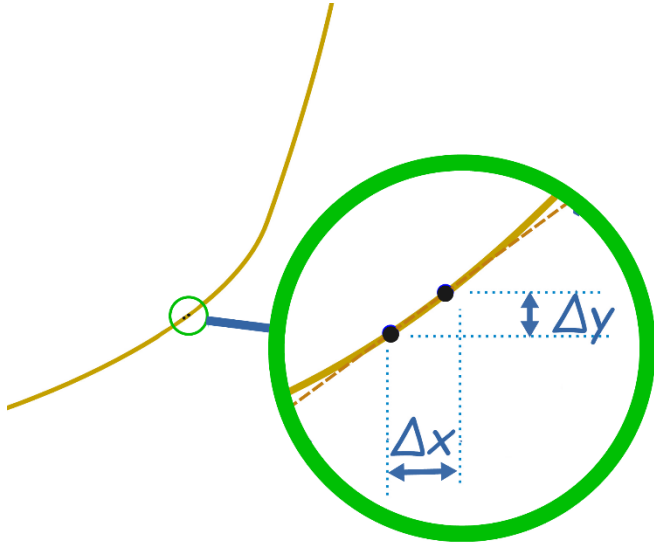
This simple idea raises further questions:

- Are all values of V permitted? Extremely high voltages might alter the resistor's properties.
- How do we combine functions if V itself is a function of other network voltages?
- Can we find an inverse function that defines how voltage depends on the current? (e.g., from $I = V/1000$ to $V = 1000I$).

Why Study Sets and Functions?

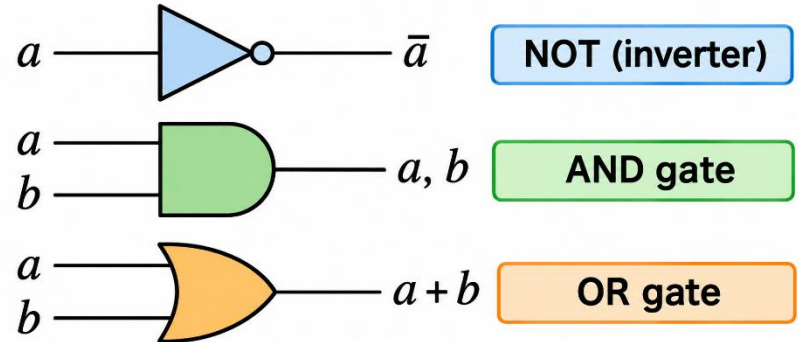
Understanding Change

A good understanding of functions is crucial for studying differentiation and integration, which analyze how functions change.



Logic and Structure

A set is a collection of objects. Studying sets helps in understanding functions and the properties of logic circuits used in digital design.



What is a Set?

A set is a collection of objects, called elements, in which order is not important and an object cannot appear twice.

' $a \in A$ ' means ' a is an element of A '.

Set A

$\{a, b, c\}$

Set B

$\{3, 4, 6, 7, 8, 9\}$

Set C

$\{Linda, Raka, Sue, Joe, Nigel, Mary\}$

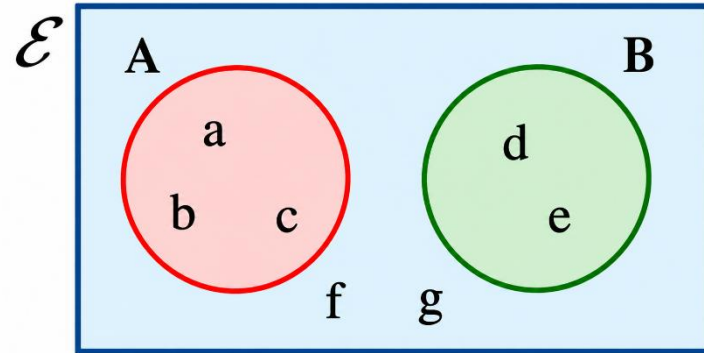
Fundamental Set Concepts

Key Sets

- Universal Set ($\mathcal{E}$): The set of all objects under consideration in a given problem.
- Empty Set ($\emptyset$ or $\{\}$): The set containing no elements.

Venn Diagrams

Sets can be represented diagrammatically using Venn diagrams. The universal set is a rectangle, and sets within it are typically circles.



Important Sets of Numbers

1 Natural Numbers ($\mathbb{N}$)

The set of counting numbers: $\{1, 2, 3, 4, 5, \dots\}$

3 Rational Numbers ($\mathbb{Q}$)

The set of numbers that can be expressed as a fraction, including integers.

2 Integers ($\mathbb{Z}$)

The set of negative and positive whole numbers and zero:

$\{\dots - 3, -2, -1, 0, 1, 2, 3 \dots\}$

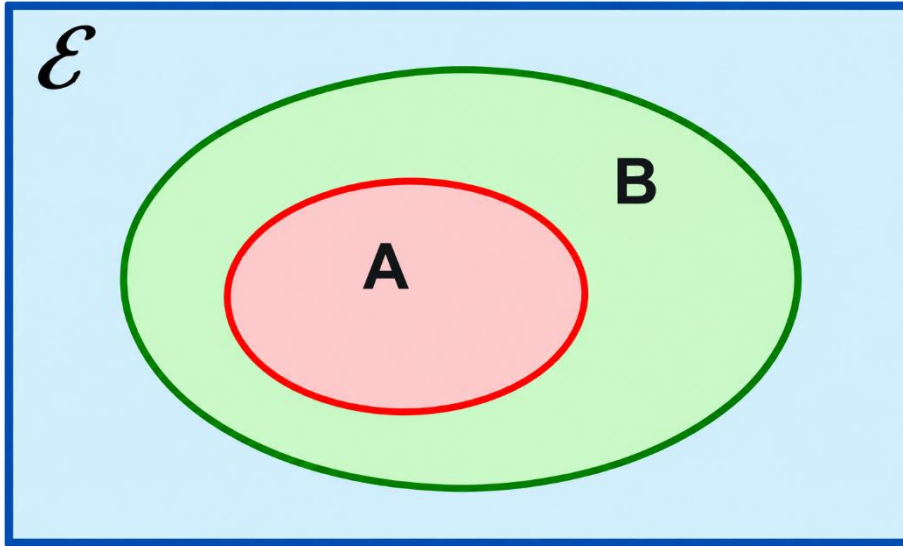
4 Real Numbers ($\mathbb{R}$)

All numbers on the number line, including rationals and irrationals.

1.3 Operations on Sets

Subset ($A \subseteq B$)

Every element of A is also an element of B.



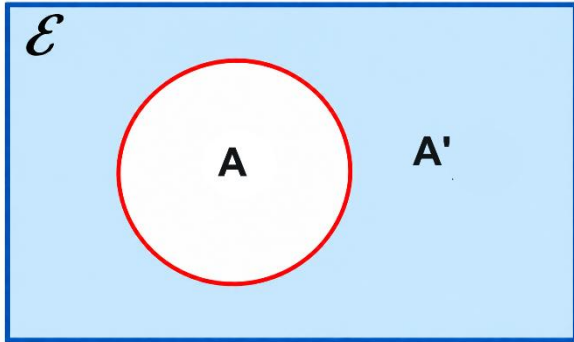
Proper Subset ($A \subset B$)

A is a subset of B, but A is not equal to B.

Set Operations

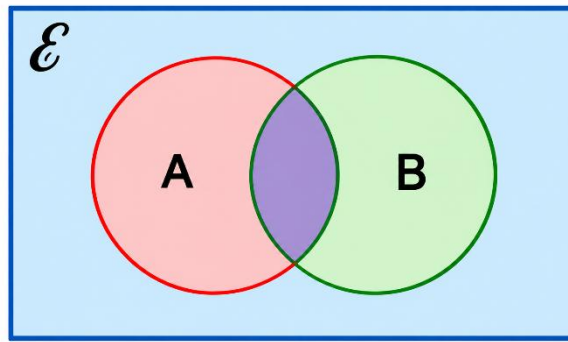
Complement (A')

Everything in the universal set that is NOT in A.



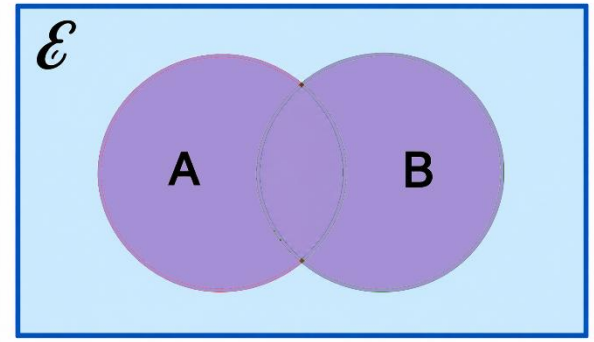
Intersection ($A \cap B$)

Elements that are in BOTH A and B.



Union ($A \cup B$)

Elements that are in A OR B OR both.



Cardinality of a Set

The number of elements in a set is called its cardinality, written as $n(A)$ or $|A|$.

- $n(\emptyset) = 0$
- $n(\{2\}) = 1$
- $n(\{a, b\}) = 2$

Example

Survey result:

100 students (S)

720 people owned a tablet (T)

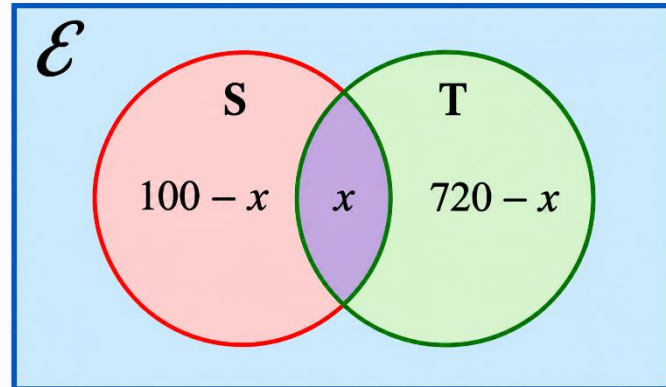
794 people owned a tablet or were students

How many students own a tablet?

We have

$$n(S \cup T) = n(S) + n(T) - n(S \cap T)$$

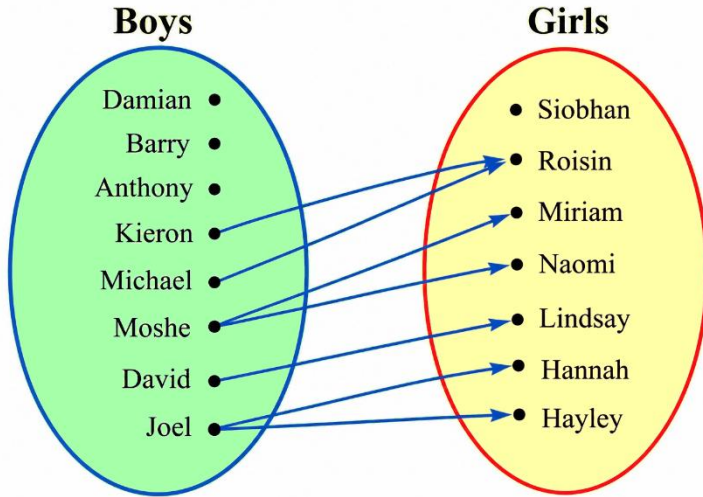
So, $794 = 100 + 720 - x$, which gives $x = 26$.



1.4 Relations and Functions

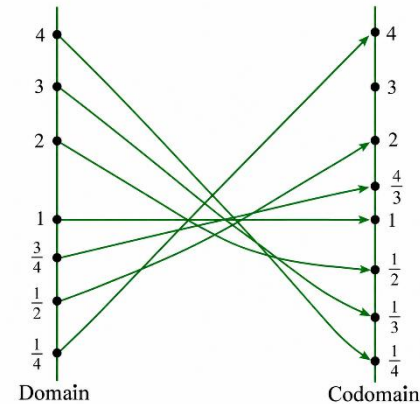
Relation

A way of pairing members of two sets. An input can be paired with zero, one, or multiple outputs, like this example of boys and their sisters.



Function

A special relation where each input value has exactly one output value. It's a reliable mathematical machine.

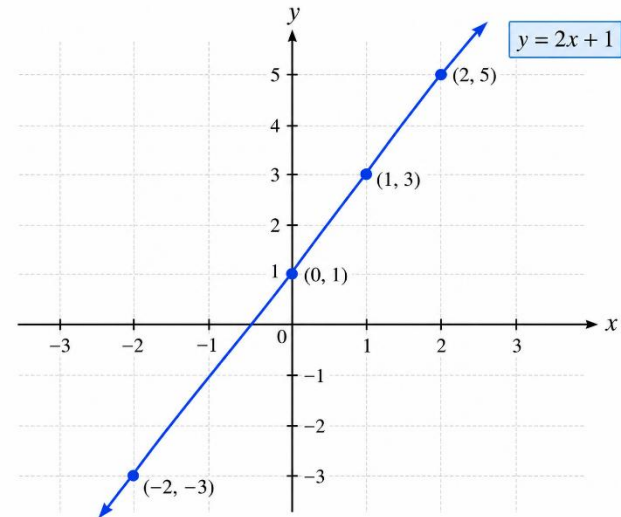


The function $1/x$ shown as a mapping diagram, $x > 0$

The Language of Functions

- Domain: The set of all possible input values.
- Codomain: A set containing all possible output values.
- Image: The specific output value for a given input.
- Notation: We write $f(x) = y$, where y is the image of x under the function f .
- Variables: x is the independent variable, and y is the dependent variable.
- Representation: functions, particularly of a real variable, are usually pictured using graphs with the domain on the horizontal axis and the codomain on the vertical axis.

Input and output values are marked by a point at the position (x, y)

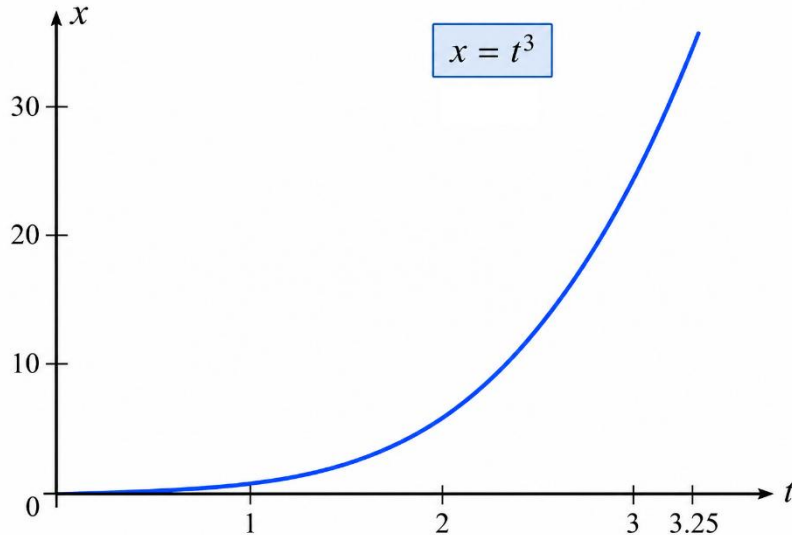


Continuous vs. Discrete Functions

Functions are used to represent signals. Analog signals are continuous, or piecewise continuous, while digital signals are discrete.

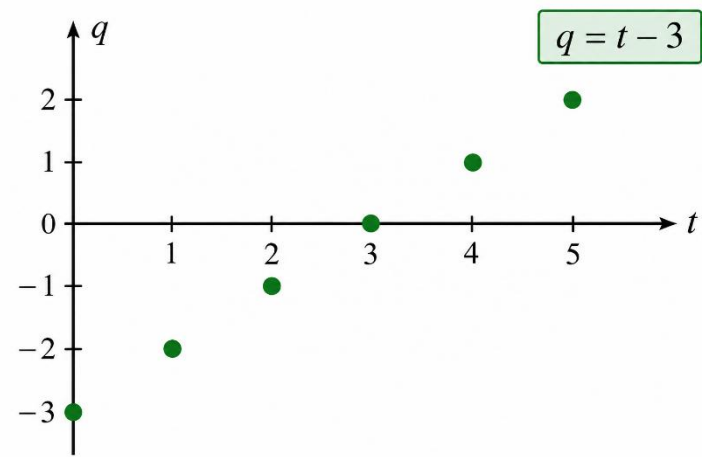
Continuous Functions

The function's graph can be drawn without lifting the pen.
Represents analog signals.



Discrete Functions

The function is defined only at specific points. Represents digital signals.



Discrete Functions, Sequences and Time Sequences

- ❖ Functions of an integer are also called sequences
- ❖ As a sequence is a function of the natural numbers and zero (or, if negative input values are allowed, the integers) there is no need to specify the input values and it is possible merely to list the output values in order. Hence the function $q_n = n - 3$ for $n \geq 0$ and n an integer can be expressed by $-3, -2, -1, 0, 1, 2, 3 \dots$
- ❖ Digital signals can be obtained by using an analog to digital (A/D) convertor on an originally continuous signal. In this case there will usually be a constant sampling interval T and the time sequence is assumed to be a function of time, $t = nT$, where n is a natural number or 0.
- ❖ Sequences can also be defined using a recurrence relation e.g. $q_n = n - 3$ can be represented by $q_{n+1} = q_n + 1$ with $q_0 = -3$ and $n \geq 0$

Undefined Function Values

Some numbers cannot be input into a function. These must be excluded from the domain. Things to look out for as values that are not allowed as function inputs are :



Division by Zero

The denominator of a fraction cannot be zero. For $f(x) = 1/x$, we must exclude $x = 0$.

Negative Square Roots

The value inside a square root cannot be negative (in real numbers). For $f(x) = \sqrt{x}$, we require $x \geq 0$.

Negative/Zero Logarithms

The value inside a logarithm must be positive. For $f(x) = \log(x)$, we require $x > 0$.

Combining Functions

Arithmetic Combinations

- **Sum:** $(f + g): x \mapsto f(x) + g(x)$
- **Difference:** $(f - g): x \mapsto f(x) - g(x)$
- **Product:** $(f \times g): x \mapsto f(x) \times g(x)$
- **Quotient:** $\left(\frac{f}{g}\right): x \mapsto \frac{f(x)}{g(x)}$ provided $g(x) \neq 0$

Composition of Functions

Composition is performing one function followed by another: a function of a function. The order matters!

Let $f(x) = 2x + 1$ and $g(x) = x^2$

$g(f(x))$ or $(g \circ f)(x)$

First f , then g :

$$g(2x + 1) = (2x + 1)^2$$

$f(g(x))$ or $(f \circ g)(x)$

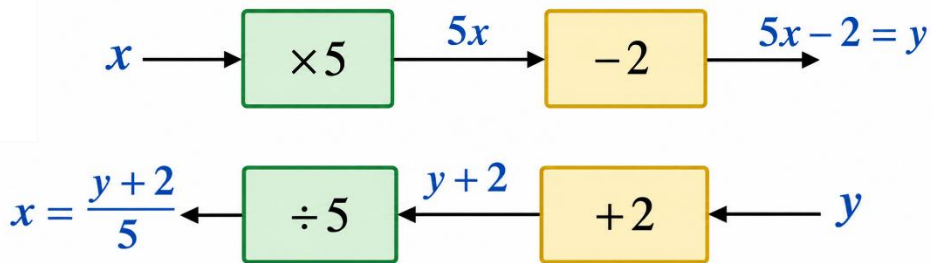
First g , then f :

$$f(x^2) = 2(x^2) + 1 = 2x^2 + 1$$

Decomposing Functions and Finding the Inverse

A composite function can be broken into different stages, each represented by its own box. The inverse of a function is a function which will take the image under the function back to its original value.

Example: $f(x) = 5x - 2$ and its inverse



Top line: $f(x) = 5x - 2$ (read from left to right)

Bottom line: the inverse function (read from right to left).

The inverse operations give that $x = \frac{y+2}{5}$.

Swap the letters, giving the inverse function as $y = \frac{x+2}{5}$

If $f^{-1}(x)$ is the inverse of $f(x)$, then

$$f^{-1}(f(x)) = x$$

$$(f^{-1} \circ f) : x \rightarrow x$$

Finding the Inverse

- Write the function as $y = f(x)$
- Rearrange the equation to make x the subject (solve for x in terms of y).
- Swap the letters x and y . The new expression for y is $f^{-1}(x)$

1.6 Summary

- Functions are used to express relationships between physical quantities.
- A set is a collection of elements.
- Key set operations are intersection ($\cap$), union ($\cup$), and complement ($\mathbf{A}'$).
- The set of all inputs to a function is the domain of the function. The set including all the outputs is called the codomain.
- Functions are special relations where each input has exactly one output.
- Analog signals are represented by continuous, or piecewise continuous, functions; digital signals by discrete functions (sequences).
- Functions can be combined via arithmetic operations or composition (function of a function).
- The inverse of a function is a function which will take the image under the function back to its original value $f^{-1}(f(x)) = x$