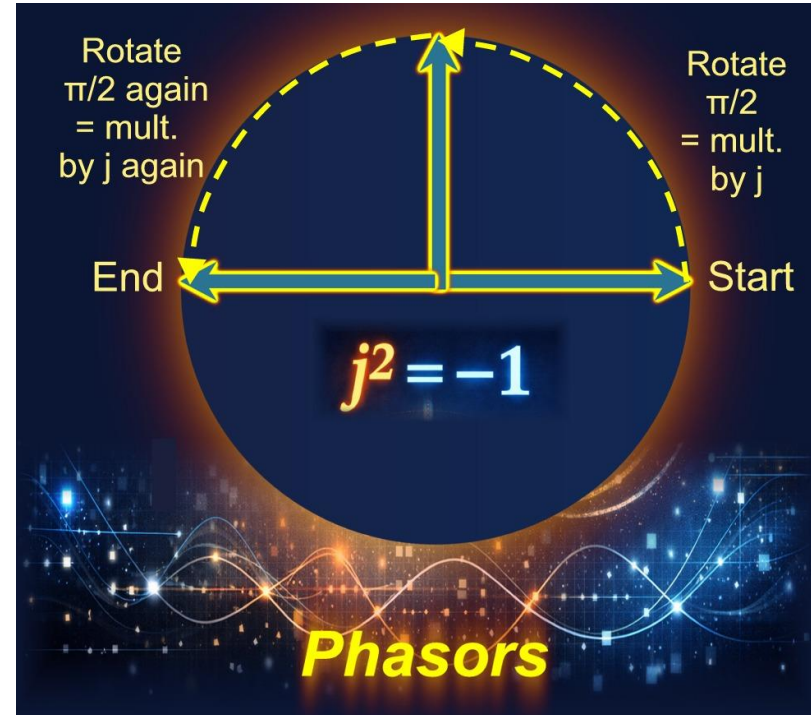




Complex Numbers



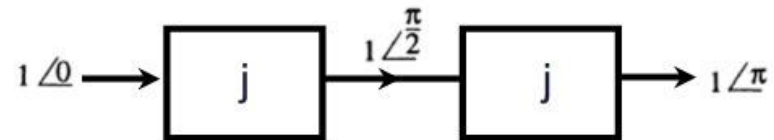
Introduction to Complex Numbers

A single frequency wave can be represented by a rotating vector called a phasor. Linear systems can alter this wave by scaling its amplitude and shifting its phase. Using complex numbers, this entire system transformation can be represented by a simple multiplication.

The Imaginary Unit, j

We define ' j ' as an operator that rotates a phasor by $\pi/2$ (90°) without changing its amplitude.

- Multiplying by j rotates a phasor by $\pi/2$.
- Multiplying by j twice ($j \times j$) rotates the phasor by π (180°), which is equivalent to multiplying by -1 .
- This leads to the fundamental definition of the imaginary unit: $j^2 = -1$.

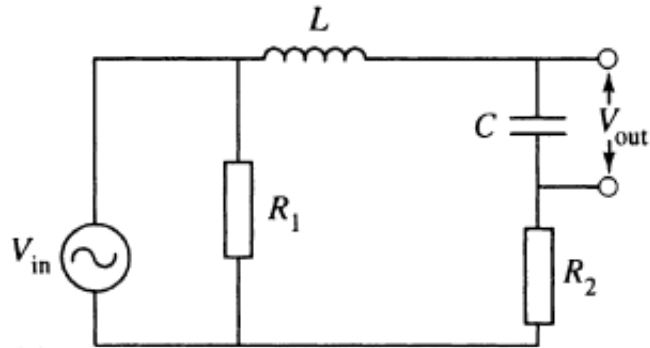


Linear Systems and Phasors

A system is often represented as a box with an input and an output. We are interested in linear systems where a single-frequency wave input produces an output at the same frequency, but with a potentially different amplitude and phase.

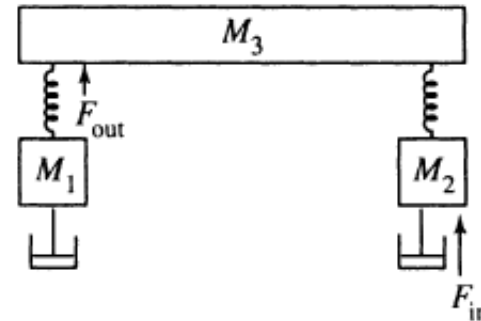
Electrical Systems

Circuits made of resistors, capacitors, and inductors are a prime example. The input and output are typically voltages.



Mechanical Systems

Equivalent mechanical systems consist of masses, springs, and dampers. The input might be an external force and the output a displacement or tension.



Representing Complex Numbers

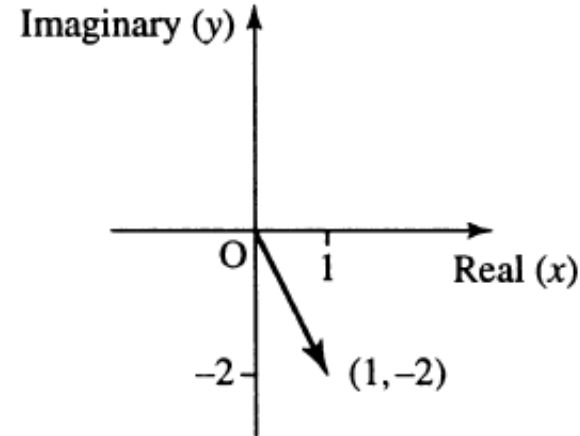
Real and Imaginary Parts

A complex number, z , is written as the sum of its real and imaginary parts: $z = x + jy$. In engineering, ' j ' is used for the imaginary unit to avoid confusion with ' i ' for current.

- x is the real part: $Re(z) = x$
- y is the imaginary part: $Im(z) = y$

The Complex Plane

Complex numbers are visualized on a 2D plane, often called an Argand diagram. The horizontal axis represents the real part, and the vertical axis represents the imaginary part.



Basic Operations with Complex Numbers

Equality

Two complex numbers are equal only if their real parts are equal and their imaginary parts are equal.

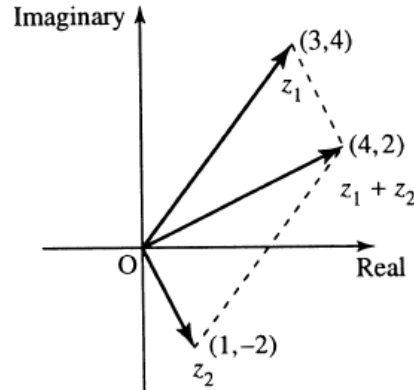
If $a + jb = c + jd$
then $a = c$ and $b = d$

Addition

Add the real and imaginary parts separately.

$$(a + jb) + (c + jd) = (a + c) + j(b + d)$$

This follows the parallelogram law for vectors.

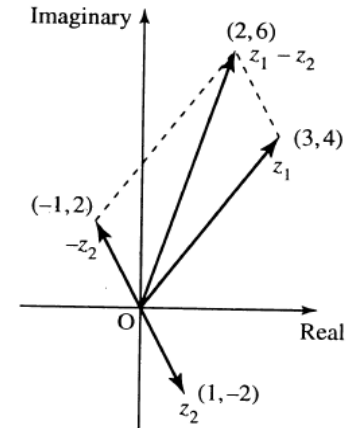


Subtraction

Subtract the real and imaginary parts separately.

$$(a + jb) - (c + jd) = (a - c) + j(b - d)$$

This is like adding the negative of the second vector.



Multiplication and Conjugates

Multiplication

Multiply complex numbers in the same way as any algebraic expression, by expanding the brackets. Finally remember that $j^2 = -1$

Example:

$$\begin{aligned} & (3 + j4)(1 - j2) \\ &= 3(1) + 3(-j2) + j4(1) + j4(-j2) \\ &= 3 - j6 + j4 - j^2 8 \\ &= 3 - j2 + 8 \quad (\text{using } j^2 = -1) \\ &= 11 - j2. \end{aligned}$$

The Complex Conjugate

The conjugate of $z = x + jy$ is $z^* = x - jy$
Geometrically, it's a reflection across the real axis.

- A number multiplied by its conjugate is always a real, non-negative number.
- $z z^* = (x + jy)(x - jy) = x^2 + y^2$
- This value is also the square of the number's magnitude (modulus): $|z|^2$

Division of Complex Numbers

To divide complex numbers, we make the denominator a real number. This is done by multiplying both the numerator and the denominator by the complex conjugate of the denominator.

Example:
$$\frac{3 + j4}{1 - j2}$$

- Step 1: Multiply top and bottom by the conjugate of the denominator ($1 + j2$)

- Fraction becomes:
$$\frac{(3 + j4)(1 + j2)}{(1 - j2)(1 + j2)}$$

- Step 2: Expand the numerator and denominator.
- Numerator: $(3 + j6 + j4 + j^2 8) = (3 + j10 - 8) = -5 + j10$
- Denominator: $1^2 + 2^2 = 1 + 4 = 5$

- Step 3: Simplify the result:
$$\frac{-5 + j10}{5} = -1 + j2$$

Solving Quadratic Equations

Complex numbers provide a way to find solutions for any quadratic equation of the form $ax^2 + bx + c = 0$

The Formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- When $b^2 - 4ac < 0$, the term inside the square root is negative, which meant, when we only considered real numbers, there were no solutions.
- With complex numbers, when $b^2 - 4ac < 0$ we can write $\sqrt{b^2 - 4ac}$ as $j\sqrt{4ac - b^2}$
- This means the solutions will be a pair of complex conjugates of the form $p \pm jq$
- Therefore, if you know one complex root of a quadratic equation with real coefficients, the other root is simply its conjugate.

Polar Form of a Complex Number

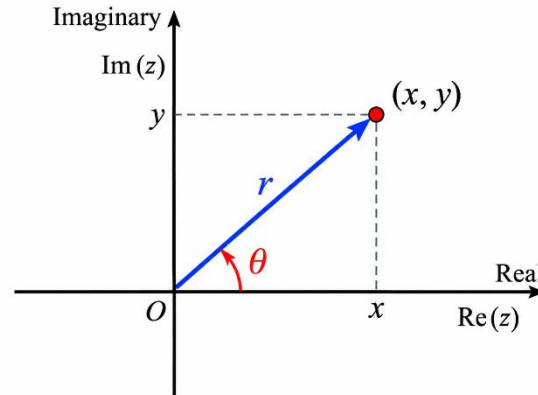
Instead of Cartesian coordinates (x, y) , we can describe a complex number using its distance from the origin (modulus, r) and its angle from the positive real axis (argument, θ). This is the polar form: $r\angle\theta$

Cartesian to Polar

- Modulus: $r = |z| = \sqrt{x^2 + y^2}$
- Argument: $\theta = \tan^{-1}(y/x)$ (with adjustment for the correct quadrant)

Polar to Cartesian

- Real Part: $x = r \cos(\theta)$
- Imaginary Part: $y = r \sin(\theta)$



Operations in Polar Form

While addition and subtraction require conversion to Cartesian form, multiplication and division become much simpler in polar form.

Multiplication

To multiply, multiply the moduli and add the arguments.

$$(r_1 \angle \theta_1) \cdot (r_2 \angle \theta_2) = (r_1 r_2) \angle (\theta_1 + \theta_2)$$

Division

To divide, divide the moduli and subtract the arguments.

$$(r_1 \angle \theta_1) / (r_2 \angle \theta_2) = (r_1 / r_2) \angle (\theta_1 - \theta_2)$$

Application: AC Circuit Analysis

Complex numbers are fundamental to analysing alternating current (AC) circuits. The relationship between voltage (V), current (I), and impedance (Z) is a complex version of Ohm's Law: $V = ZI$



Resistor

Impedance $Z = R$

It's a purely real value representing resistance.



Capacitor

Impedance $Z = 1/(j\omega C) = -j/(\omega C)$

It's purely imaginary and called reactance. ω is the angular frequency.



Inductor

Impedance $Z = j\omega L$

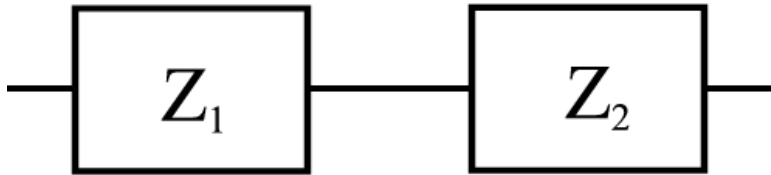
It's also a purely imaginary reactance.

Combining Impedances

Series Circuits

For components in series, the total impedance is the sum of the individual impedances.

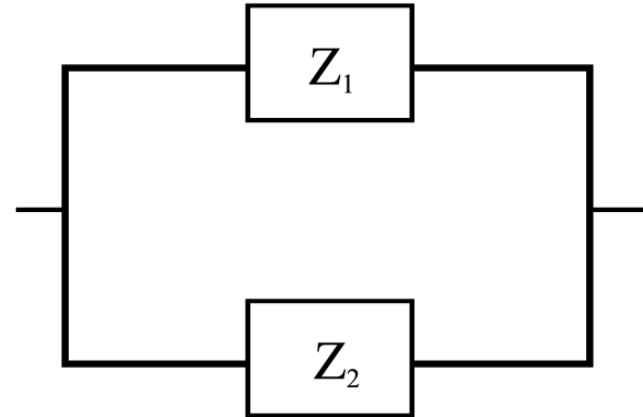
$$Z_{total} = Z_1 + Z_2$$



Parallel Circuits

For parallel components.

$$\begin{aligned} 1/Z_{total} &= 1/Z_1 + 1/Z_2 \\ \text{or} \quad Y_{total} &= Y_1 + Y_2 \end{aligned}$$



Circular Motion, Polar and Exponential Form

The Rotating Vector

A point moving in a circle with radius r at a constant angular velocity ω can be described by:

$$z = r \cos(\omega t) + jr \sin(\omega t)$$

z is a solution to the differential equation

$$\frac{d^2 z}{dt^2} = -\omega^2 z$$

which defines simple harmonic motion.

Exponential form

We can show that $z = r e^{j\omega t}$ is also a solution to the differential equation

$$\frac{d^2 z}{dt^2} = -\omega^2 z$$

with the same initial conditions: $z = r$ and velocity $dz/dt = jr\omega$, at $t = 0$

So,

$$r e^{j\omega t} = r \cos(\omega t) + jr \sin(\omega t)$$

This shows the equivalence of the polar form of a complex number and the exponential form. Replacing ωt by θ , it allows us to write any complex number in exponential form:

$$z = r e^{j\theta}$$

$$\text{Where } e^{j\theta} = \cos(\theta) + j \sin(\theta)$$

The Exponential Form

The exponential form $z = r e^{j\theta}$ simplifies complex number operations as we can use the standard rules of exponents.

1 Multiplication

$$r_1 e^{j\theta_1} r_2 e^{j\theta_2} = r_1 r_2 e^{j(\theta_1 + \theta_2)}$$

2 Division

$$\frac{r_1 e^{j\theta_1}}{r_2 e^{j\theta_2}} = \frac{r_1}{r_2} e^{j(\theta_1 - \theta_2)}$$

3 Powers

$$(r e^{j\theta})^n = r^n e^{jn\theta}$$

This can be used to show De Moivre's theorem

$$(r(\cos(\theta) + j \sin(\theta)))^n = r^n (\cos(n\theta) + j \sin(n\theta))$$

So

$$(\cos(\theta) + j \sin(\theta))^n = \cos(n\theta) + j \sin(n\theta)$$

Trigonometric and Hyperbolic Functions

Using $e^{j\theta} = \cos(\theta) + j \sin(\theta)$ we can find expressions linking trigonometric and hyperbolic functions.

Trigonometric Expressions

$$\cos(\theta) = \frac{1}{2}(e^{j\theta} + e^{-j\theta})$$

$$\sin(\theta) = \frac{1}{2j}(e^{j\theta} - e^{-j\theta})$$

$$\tan(\theta) = \frac{1}{j} \left(\frac{e^{j\theta} - e^{-j\theta}}{e^{j\theta} + e^{-j\theta}} \right)$$

Link to Hyperbolic Functions

- $\cos(j\theta) = \cosh(\theta)$
- $\sin(j\theta) = j \sinh(\theta)$
- $\tan(j\theta) = j \tanh(\theta)$

Solving Equations: The Roots of Unity

The exponential form is invaluable for finding the n -th roots of a complex number, such as solving the equation $z^n = 1$, known as finding the ' n -th roots of unity'.

The Method

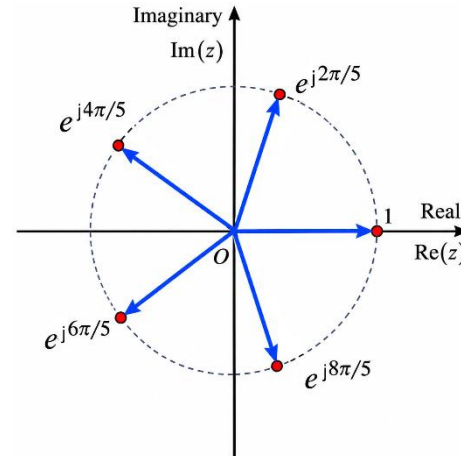
- Write the number in its general exponential form, acknowledging that angles repeat every 2π . For 1, this is $e^{j2\pi N}$, where N is an integer.
- The equation becomes $z^n = e^{j2\pi N}$
- Take the n -th root of both sides:

$$z = e^{\frac{j2\pi N}{n}}$$

- Let $N = 0, 1, 2, \dots, n-1$ to find the n distinct roots.

The Result

The n roots are always evenly spaced around a circle in the complex plane, e.g. the 5th roots of unity below.



Summary

Complex Numbers

- $j^2 = -1$
- Cartesian form: $z = x + jy$
- Polar form: $z = r\angle\theta$
- Exponential form: $z = re^{j\theta}$

Operations

- Add/Subtract in Cartesian
- Multiply/Divide in Polar/Exponential
- Use conjugate for division
- Powers: $(re^{j\theta})^n = r^n e^{jn\theta}$

Applications

- Solving quadratics
- AC circuit analysis ($V = ZI$)
- Describing circular motion & waves

Key Formulas

Euler's formula: $e^{j\theta} = \cos(\theta) + j \sin(\theta)$

De Moivre's Theorem: $(\cos(\theta) + j \sin(\theta))^n = \cos(n\theta) + j \sin(n\theta)$