Background Mathematics Notes

Mathematics for Electrical Engineering and Computing

Mary Attenborough

Chapter 1 Numbers and Operations

1.1 INTRODUCTION

Numbers and their operations are used to represent real-life situations. For example, a typical addition problem could be: there were 3 cows in a field and they were joined by 5 cows, making 8 in all. The use of numbers allows us to think of different situations as represented by the same mathematical problem, $3 + 5 = 8$. The arithmetic remains the same whether the animals are cows, sheep or pigs. Representing a real-life situation by a mathematical one is called mathematical modelling.

Thinking of a simple real-life problem can often help to work out what the result of an operation should be. Supposing we have forgotten how to divide fractions and come across the problem:

$$2 \div \frac{1}{3}$$

We could invent a simple problem to help out in that situation. 'There were two bars of chocolate and each one was divided into thirds. How many pieces are there? '. We could then clearly see that the answer should be six. However, this approach will not always help. For instance, it is difficult to think of a simple everyday problem that would lead to the multiplication:

(-3)(-6)= ?

In these cases, where the result is not obvious, we need to remember established rules for the outcome. In this chapter we shall examine the rules of the operations of addition, subtraction, multiplication and division on real numbers. The real numbers are all those numbers, including positive and negative whole numbers and fractional and decimal numbers that represent points on the real number line. When we draw a number line we usually mark only the whole numbers. This does not mean that the others do not concern us. The line is assumed to continue indefinitely in both directions, therefore representing all possible real numbers. A real number line is shown in Figure 1.1.
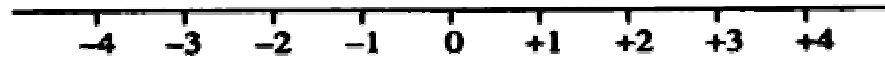
Figure 1.1 A real number line.

1.2 Algebra: using letters for numbers

There are a number of reasons that we might use a letter to represent a number. One situation is for a scientific formula, for example Ohm's Law, V = IR, which gives the voltage across a pure resistor given the current and resistance. This formula is a statement that is supposed to be true in all relevant situations. Hence the letters are used to represent any physically possible values for the quantities, voltage, current and resistance.

Letters can be used in the same way to state mathematical laws. For instance, you may wish to state that it does not matter whether you swap the order of an addition: the result will be the same. This can be expressed by the statement:

For any two numbers *a* and *b*, *a* + *b* = *b* + *a*

This statement sums up all possibilities for adding any two numbers and *a* and *b* could be replaced by any numbers to giving as example:

$$3 + 5 = 5 + 3$$

$$8 + 7 = 7 + 8$$

$$1 + 2 = 2 + 1$$

$$2.1 + 8.62 = 8.62 + 2.1$$

a + *b* = *b* + *a* is a generalisation of all these examples.

Algebra is the study of the rules of operations. It involves using letters because we need to generalise the rules to all possible values. In this chapter we will only concern ourselves with the algebra of numbers. The algebra of sets or matrices, for instance, could result in different rules. We need to decide on the rules and be able to use them in practice. For an engineer the important thing is to recognise that rules of manipulating numbers and their operations exist, and that the rules must not be broken.

An operation on numbers is a way of combining two numbers to give a single number. However we often need to write down expressions that involve more than a single operation. For instance the expression 3 + 4 + 5 involves two additions, while the expression 3(*x* + 4) involves one addition and one multiplication. When more than two

numbers are to be combined in this way the order in which the operations are performed may be important. Brackets can be used to indicate which operation should be performed first. However in order to simplify expressions involving more than one operation we need to know all the rules concerning which operation should be done first and how we can change the order of performing operations.

Before we go on to discuss the rules of numbers and their operations we should be aware that we shall be using letters to represent all possible numbers. This is different from using a letter to represent some unknown quantity which will take only a certain number of fixed values. For instance, $x + 3 = 5$ is called an equation. It is only true for $x = 2$ and is not true for all possible values of x. A short study of equations is given in Chapter 3.

Some short hand is used in expressions. The '.' or x indicating multiplication is often left out, particularly in an expression involving letters; i.e. ab is taken to mean $a.b$ or $a \times b$.

1.3 Rules for Addition and Multiplication

Some of the rules of numbers have special names, and because they can recur in other algebras we will list these here. These basic rules can be used to establish the every day rules of manipulating expressions. Here a, b and c can be any numbers.

$a + b = b + a$	$a.b = b.a$	Commutative laws
$(a + b) + c = a + (b + c)$	$(a.b).c = a.(b.c)$	Associative laws
$a + 0 = a$	$a.1 = a$	Identity laws
$a.(b + c) = a.b + a.c$		Distributive law
$a.0 = 0$		Multiplication by zero

The use of these laws becomes automatic with practice. The practical rules that we use are as follows:

1. In an expression involving only multiplication or addition, terms can be swapped around to any order you like. This can aid calculation:

$$2.3.5 = 2.5.3 = 10.3 = 30$$

$$2 + 5 + 8 + 5 = 2 + 8 + 5 + 5 = 10 + 10 = 20$$

This rule comes from the commutative and associative laws.

2. Adding 0 to any number has no effect. Also, multiplying by 1 leaves numbers unchanged. The number 0 is called the additive identity and 1 the multiplicative identity.
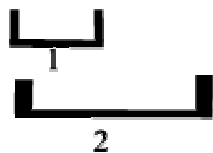
3. The distributive law is the rule used when expanding expressions.

$$a(b + c) = ab + ac$$
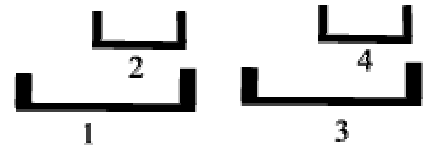
If two bracketed terms are multiplied together then the law is used repeatedly to expand the expression.

Example 1.1 Expand $(x + y)(2x + 3y)$.

SOLUTION First, expand the second bracket

$$(x+y)(2x+3y)=(x+y)2x+(x+y)3y$$

and now expand again

$$(x+y)2x+(x+y)3y=x2x+y2x+x3y+y3y$$

Use the commutative law to swap the order of the multiplications and the fact that $x.x = x^2$ to give

$$x2x + y2x + x3y + y3y = 2x^2 + 2xy + 3xy + 3y^2$$

Collecting together common terms gives

$$2x^2 + 5xy + 3y^2$$

This process can be shortened by directly multiplying out the brackets, remembering that each term in the first bracket must be multiplied by each term in the second bracket:

$$(x+y)(2x+3y)=(x+y)2x+(x+y)3y$$

Continuing as before this simplifies to

$$2x^2 + 5xy + 3y^2$$

If we swap the sides of the law so that it reads

$$a.b + a.c = a.(b + c)$$

then it is the rule which allows us to factorize, and we have taken out a common factor of a.

Example 1.2. Factorize $3(2x + 1) + x(2x + 1)$

SOLUTION There is a common factor of $(2x+1)$:

$$3(2x + 1) + x(2x + 1) = (3 + x)(2x + 1)$$

4. If more than one operation is involved in an expression brackets, (), should be used to indicate the order in which they are performed. There are two reasons for exceptions to this requirement:

(i) The associative laws mean that any number of additions, or any number of multiplications can be performed without order being important.

ii) There is a convention used to specify order. The conventional order is brackets, division, multiplication, subtraction and addition.

Example 1.3 Find 6x2x3x4 + 5x6x3 - 3x(1+2)/6

SOLUTION

$$6{\times}2{\times}3{\times}4 + 5{\times}6{\times}3 - 3{\times}(1+2)/6 = 6{\times}2{\times}3{\times}4 + 5{\times}6{\times}3 - 3{\times}3/6$$

(performing the operation in the brackets first)

$$6{\times}2{\times}3{\times}4 + 5{\times}6{\times}3 - 3{\times}3/6 = 6{\times}2{\times}3{\times}4 + 5{\times}6{\times}3 - 3{\times}0.5$$

(performing the division next)

$$6{\times}2{\times}3{\times}4 + 5{\times}6{\times}3 - 3{\times}0.5 = 144 + 90 - 1.5$$

(performing the multiplications)

$$144 + 90 - 1.5 = 144{+}88.5$$

(performing the addition)

$$144{+}88.5 = 232.5$$

(performing the subtraction)

Hence $6{\times}2{\times}3{\times}4 + 5{\times}6{\times}3 - 3{\times}(1{+}2)/6 = 232.5$

Example 1.4 Remove brackets where possible

$$(a/(b/c)) - ((pq)(rt)) + (a(b{+}c))$$

SOLUTION. Using the convention

$$(a/(b/c)) - ((pq)(rt)) + (a(b{+}c)) = a/(b/c) - pqrt + a(b{+}c)$$

The brackets are still needed in the first term to indicate that the division b/c should be performed first and a should then be divided by the result of b/c. The brackets are needed in the term $a(b{+}c)$ to indicate that $b{+}c$ should be performed before multiplying the result by a. All other brackets may be left out, as the order is either unimportant or is specified by the convention.

If in doubt use brackets to indicate order as they never do any harm.

Before we look at the rules for subtraction we will look more closely at addition and multiplication in particular, in order to understand the behaviour of negative numbers.

1.4 POSITIVE AND NEGATIVE NUMBERS

Every number, except 0, is either positive or negative. This is indicated by placing a sign in front of the number. Positive numbers may have their sign omitted. On the number line as shown in Figure 1.1, numbers to the left of 0 are negative numbers and numbers to the right are positive numbers. Here the symbols + and - are being used in a

different way from that in Section 1.3. In Section 1.3 the symbol + is an operation, an order to perform the task of addition. Similarly, - can be used as an order to perform the task of subtraction. In this section we are using the same symbols to indicate whether the number is positive or negative. To stress this distinction for the moment we shall indicate the sign of a number by placing it slightly raised in front of the number like this: $^-3$, $^+2$.

We could explain how to do addition by using the number line. Start at the position on the number line representing the first number and move to the right or left depending on whether you are adding a positive or negative number. This is illustrated in Fig. 1.2. How do we perform $^+3 + ^-2$?
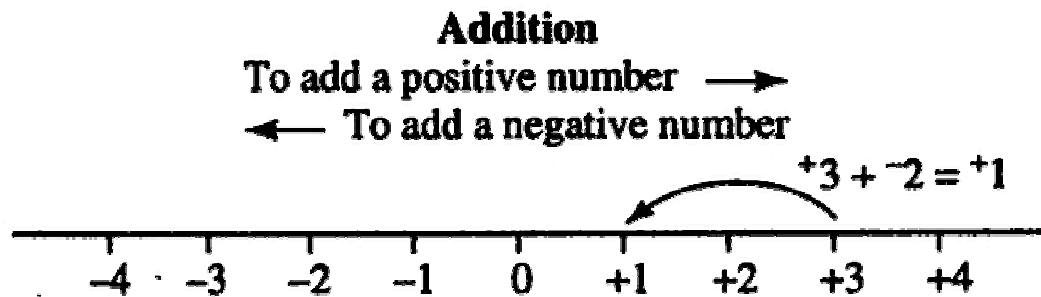
Figure 1.2 Performing addition on a number line

To add $^-2$ we move 2 units to the left. Hence

$$^+3 + ^-2 = ^+1$$

$$^-3 + ^-2 = ^-5$$

$$^+1 + ^-4 = ^-3$$

$$^+1 + ^+2 = ^+3$$

This has given us a method of adding all numbers, positive and negative. What about multiplication of positive and negative numbers? First we decide on the result of multiplying any number by $^-1$ and then use that to establish how to multiply any two numbers. One way of thinking of multiplication is as repeated addition:

$$^+3 \cdot ^+2 = ^+2 + ^+2 + ^+2 = ^+6$$

What is $^+3 \cdot ^-1$? Using repeated addition we get

$$^+3 \cdot ^-1 = ^-1 + ^-1 + ^-1 = ^-3$$

As $^+3$. $^-1$ = $^-1$. $^+3$ (the commutative law), then $^-1.^+3 = ^-3$

Multiplication by $^-1$ changes the sign of the number and is called negating the number. Thus multiplication by $^-1$ has the effect of reflecting the position of the number on the number line about 0. Fold the line at 0 and any number will meet its negated value. This is illustrated in Fig. 1.3
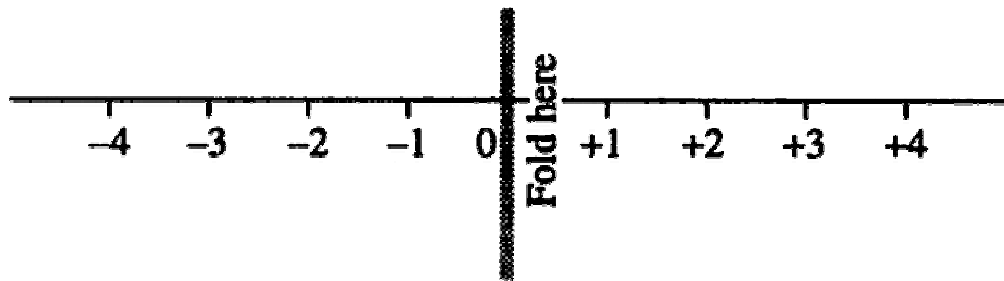
Figure 1.3 A number is negated (multiplied by $^-1$) by reflecting it about 0. Fold the line at 0 and any number will meet its negated value.

Example 1.5

$$^-1 . ^+3 = ^-3 \qquad\qquad ^-1 . ^-3 = ^+3$$

$$^-1 . ^-2 = ^+2 \qquad\qquad ^-1 . ^-1 = ^+1$$

A shorthand for $-1.x$ is $-x$, where x can take any value. Thus x negated is $-x$.

A negative number can be treated as -1 times its positive value and this is used when performing multiplication. The effect of the signs is considered last after the multiplication of the positive numbers has been performed:

$$^-5. ^-3 = ^-1.5. ^-1.3$$

$$= ^-1. ^-1.5.3 \text{ (rearranging the terms)}$$

$$= ^-1 \,^-1.15$$

$$= -1. ^-15 \text{ (negating 15)}$$

$$= 15 \text{ (negating -15)}$$

Multiplying by $^-1$ twice has the effect of multiplying by $^+1$. Thus we have a new rule governing multiplication of negative numbers, giving: **'two minuses make a plus'.**

We can extend this idea by noticing that successive multiplication by -1 repeatedly flips the sign from positive to negative.

$$^-1.^+3 = ^-3$$

$$^-1.^-1.^+3 = ^+3$$

$$^-1.^-1.^-1.^+3 = ^-3$$

$$^-1.^-1.^-1.^-1.^+3 = ^+3$$

To multiply a string of numbers together, count the number of negative numbers. If it is even, the result is positive; if it is odd the result is negative.

Example 1.6

$$^+3.^-2.^-1.^-5 = ^-30 \qquad \text{(3 minuses)}$$

$$^-2.^+6.^-8.^+1 = ^+96 \qquad \text{(2 minuses)}$$

1.5 RULES FOR SUBTRACTION

Subtraction is the opposite (more mathematically called the inverse) of addition. Subtraction can be performed on a number line by starting with the first number and then moving to the left or right, depending on whether you are subtracting a positive or a negative number, that is by moving in the opposite direction to addition. This is shown in Fig. 1.4
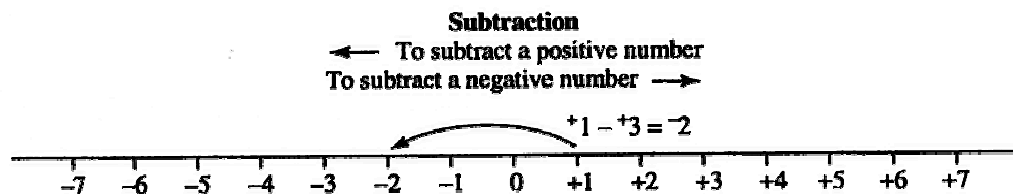
Figure 1.4 Subtraction

Compare the following two problems. First the addition $^-2 + ^-3 = ?$, which is pictured in Figure 1.5(a) and the subtraction $^-2 - ^+3 = ?$, which is pictured in Figure 1.5(b). We can see that they both result in $^-5$. This example illustrates that subtraction of any number is the same as adding its negated value. This rule is true because subtraction is the inverse of addition, and it can be expressed by

$$a - b = a + (-b) \qquad\qquad (1.1)$$

where *-b* is the negated value of *b*.

Subtraction
← To subtract a positive number
To subtract a negative number →

$^+1 - {}^+3 = {}^-2$

-7 -6 -5 -4 -3 -2 -1 0 +1 +2 +3 +4 +5 +6 +7

Figure 1.4 Subtraction

Addition
To add a positive number →
← To add a negative number

(a)

$^-2 + {}^-3 = {}^-5$

-7 -6 -5 -4 -3 -2 -1 0 +1 +2 +3 +4 +5 +6 +7

Subtraction
← To subtract a positive number
To subtract a negative number →

(b)

$^-2 - {}^+3 = {}^-5$

-7 -6 -5 -4 -3 -2 -1 0 +1 +2 +3 +4 +5 +6 +7
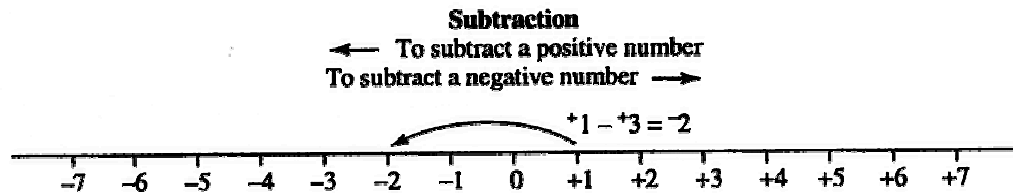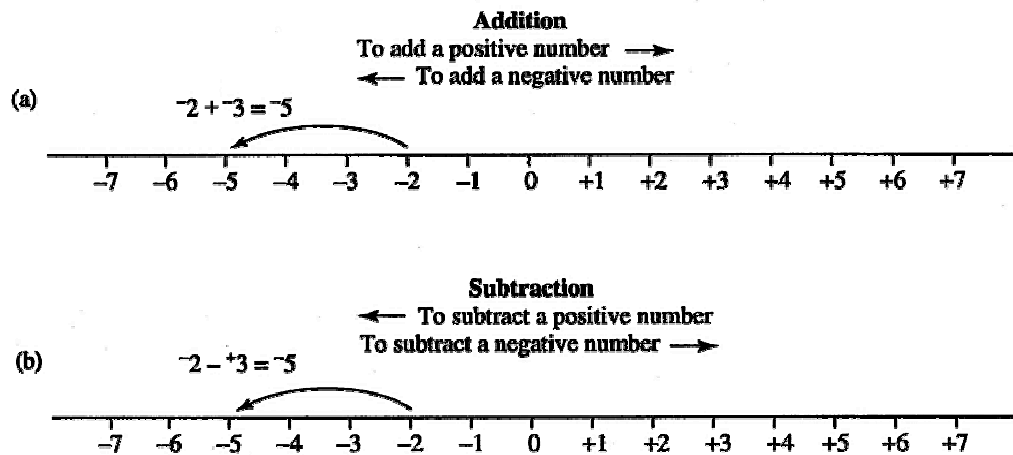
Figure 1.5 (a) The addition $^-2 + {}^-3 =$? and (b) the subtraction $^-2 - {}^+3 = ?$

The negated value of *b* can also be called its additive inverse because if you add *b* to anything and then add *-b* you get back to where you started. In other words adding *b* followed by adding *-b* results in adding 0 because b + (-b) = 0

If we substitute a negative value for b into Eq. (1.1), for instance b = $^-3$, then as the negated value of $^-3$ is $^+3$ we find

$$a - {}^-3 = a + {}^+3$$

This result can be reproduced for any negative value of b and leads to a second version of the 'two minuses make a plus' rule.

The relationship between addition and subtraction is very convenient because it allows us to replace any subtraction by an addition. We deal with any minus signs, as in the expression *a - b*, by rewriting them as *a + (-b)*. Then we are able to rearrange expressions using the rules of addition as before. This rule can be summed up as: 'attach the sign to the number it is in front of before shuffling around'.

For example:

$$a + b - c = a - c + b = - c + a + b$$

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Here we have treated *-c* as +(*-c*).

The only remaining problem with subtraction is how to cope with a minus sign in front of a bracket. This is treated as a ⁻1 multiplying the whole bracket; for example:

$$2 - (5 - 2) = 2 + {}^{-}1.(5 + {}^{-}2)$$

$$= {}^{+}2 + {}^{-}5 + {}^{+}2$$

$$= {}^{-}1$$

As we have now established the link between negated numbers and subtraction we can drop the need to raise the sign to indicate positive and negative numbers.

1.6 Division

Just as there is a very close relationship between addition and subtraction there is, of course, a similar relationship between multiplication and division.

Multiplication of whole numbers can be interpreted as repeated addition. The rectangle in Fig. 1.6 has an area that can be found by adding each of the areas of the horizontal strips. This gives an area of $4 \times 3 = 12$ square units.
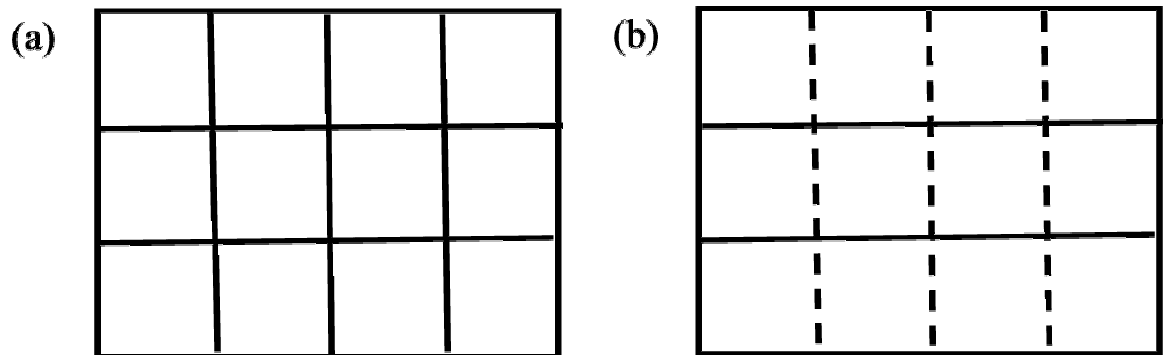
Figure 1.6 (a) A 4 by 3 rectangle has an area of 4 x 3 = 12 square units

Division is then the inverse process, and can be thought of by the problem, 'how may horizontal strips of area 3 square units are there in a rectangle of area 12 square units?'. Here we are thinking of division arising from the problem of finding the value x such that:

$$x \times 3 = 12$$

giving the solution as $12 / 3 = 4$

This reasoning tells us that division is almost the inverse operation to multiplication. It is only 'almost' because of the problem with zero. Division by zero is not defined. If we think of division as reversing the process of multiplication then it is clear that there is no way of reversing multiplication by zero. Multiplication by 0 always gives 0, for instance

$$0 \times 6 = 0$$

$$0 \times 12 = 0$$

$$0 \times 9 = 0$$

All of these give the same result and therefore given the problem $0.x = 0$ we cannot determine the value of x. Hence division by zero is not defined.

For all non-zero numbers, instead of performing division we can perform an equivalent multiplication. This is a familiar idea in the language we use to describe division. Division of a cake into two pieces is to find a half of the cake. Division by 3 is the same as finding one third of something. Notice that 2 and $\frac{1}{2}$ are **reciprocals** of one another, as are 3 and $\frac{1}{3}$. To find the reciprocal of a number write it as a fraction and turn the fraction upside down. 2 can be written as 2/1; therefore its reciprocal is $\frac{1}{2}$. The reciprocal of x is the number $1/x$.

The reciprocal of a number is also called its multiplicative inverse because if you multiply anything by x and then by $1/x$ you get back to where you first started. In other words, if you multiply by x and then by $1/x$ you have multiplied by 1 overall because

$$x.\frac{1}{x} = 1$$

In many situations, division can be replaced by multiplication using the multiplicative inverse. However, all the rules of division cannot be so conveniently discarded, as we did for subtraction, since we want to be able to use fractional expressions. We will concern ourselves more with coping with fractional expressions in the next chapter.

Note the following properties *not* obeyed by division:

a/b ≠ b/a   division is not commutative

(a/b)/c ≠ a/(b/c)   division is not associative

c/(a + b) ≠ c/a + c/b   division is not left distributive

However, division does obey the right distributive law:

$$(a + b)/c = a/c + b/c$$

1.7 EQUALITY OF EXPRESSIONS

If we state that two expressions are equal then in general we mean they are equal for all possible values; that is, the letters should be able to be replaced by any number.

As there are an unlimited number of numbers it is not a good idea to start substituting in order to show that the expressions are equal, this would require an infinite amount of work. To show that two expressions are equal we must use the established rules for the behaviour of numbers and their operations.

There are three possible ways to go about showing that some equality is correct:

1. Start with the left hand side (LHS) of the equals sign and try to manipulate it into the same form as right-hand side (RHS).

2. Start with the right hand side and try to get it to the same form as the left hand side.

3. Rearrange both sides of the statement to show that they are both equal to a third expression.

If you suspect, however, that some equality is false, there is a quick method to show this: find any numbers which, when substituted for the letters, lead to a false statement.

Example 1.7 Show that $2(a + b) - 2a = 2b$

SOLUTION. Remove the brackets on the left hand side of the equality:

$$2(a + b) - 2a = 2a + 2b - 2a = 2b$$

This is now equal to the RHS of the original statement.

Example 1.8 Show that $a.b + c = a.(b + c)$ is a false statement.

SOLUTION. Substitute any values for a, b and c for instance, $a = 2$, $b = 3$, $c = 1$, this gives:

LHS: $a.b + c = 2.3 + 1 = 7$

RHS: $a.(b + c) = 2.(3 + 1) = 2.4 = 8$

Since the RHS is different from the LHS then the statement is false.

1.8 SUMMARY

1. The order of operations follows brackets, division, multiplication, subtraction and addition.

2. In expressions involving only addition or only multiplication the order is not important.

3. The distributive law

$$a(b+c) = ab + bc$$

is used to expand expressions or to take out a factor.

4. To negate a number, flip it over to the opposite side of 0 on the number line, i.e. multiply by -1. x and $-x$ are additive inverses, and they add up to 0.

5. To avoid any problems with subtraction convert to addition: 'attach the sign to the number it is in front of before shuffling around'.

6. Turn a number upside down to find its reciprocal (also called its multiplicative inverse). That is x and $1/x$ are multiplicative inverses and they multiply together to give 1.

7. Two minuses make a plus:

$$a\text{-}(\text{-}b) = a + b$$

$$(\text{-}a).(\text{-}b) = ab$$

8. Division by zero is undefined.

9. To prove some statement involving an equality use the established rules of how numbers behave to rearrange one or both sides.

10. To disprove some statement involving an equality substitute particular numbers to give a false statement.

1.9 EXERCISES

1.1 For each part use letters to represent a rule that generalises all of the given inequalities. Do you think your rule is true?

a) $3 \times 5 = 5 \times 3$, $4 \times 6 = 6 \times 4$, $3 \times 7 = 7 \times 3$

b) $2 \times (4 + 2) = 2 \times 4 + 2 \times 2$ $\qquad$ $3 \times (4 + 6) = 3 \times 4 + 3 \times 6$

c) $3 + (4 + 2) = (3 + 4) + 2$

d) $1/(1/3) = 3$, $1/(1/4) = 4$, $1/(1/5) = 5$

1.2 Use the commutative, associative, and distributive laws only to justify the following statement:

$$a((c+ d)b) = abc + adb$$

1.3 Give the negated values of the following:

(a) $^+ 5$ (b) $^-10$ (b) $-x$ (d) $\dfrac{1}{-x}$

1.4 Simplify the following:

(a) $a - 5b - (3b - 5a) - 3(a + b)$

(b) $-c + a - 2b + (5 - (3b - 2a)).(-2)$

1.5 Does subtraction obey they following laws?

a) $(a - b) - c = a - (b - c)$ associative law

b) $a - 0 = 0 - a = a$ identity law

c) $a(b - c) = ab - ac$ distributive law

1.6 Does division have a right identity i.e. is there a number x such that for all possible values of a then

a/x = a

1.7 Does division have a left identity, i.e. is there a number x such that for all possible values of a then

x/a = a

1.8 What is the reciprocal of $\dfrac{1}{x}$?

1.9 Calculate

(a) $3 - 4 \times 2 + 5(4 + 6)/2$

(b) $7 - 3 - 2 + 4(3 + 7) - (5 - 2)$

(c) $(4 - 2)(3 + 2)(3 \times 2)$

(d) (5 - 3)/(-2 + 4) - 7

1.10 Remove the brackets and combine like terms

(a) $(p + 2q) - (p - 3q)$

(b) $3(2x - y) - 2(3x + y)$

(c) $(y + 3)(y + 2)$

(d) $2(a + b)^2 + 3(a + b)^2$

1.11 State whether the following are true or false and justify your answer, where p, q, x and y can take any values.

(a) $x(5 - y) - 3(5 - y) = (5 - y)(x - 3)$

(b) $2p - 6 + pq - 3q = (p - 3)(2 + q)$

(c) $p^2 - 2p + 3 = -p^2 + 3$

(d) $(p^2-1)/(p + 1) = p - 1$ where $p \neq -1$

(e) $5pq^2 - 2p^2q + pq = -3p(p + q)$

(f) $-4p - 6 \times 2q + 8(p - 3 \times 3q) = 4p - 9 \times 9q$

1.12 Give the additive inverse of each of the following:

(a) - 3   (b) $\dfrac{1}{7}$ (c) - a   (d) $x - y$ (e) -(x - y)

1.13 Give the multiplicative inverse of each of the following:

(a) 0.25   (b) –0·25

(c) $\frac{9}{14}$ (d) a

(e) $\dfrac{1}{a}$ (f) a - 4

(g) a^2

1.14 Simplify

(a) $3(2x - y) - 2(3x + y)$

(b) $a(a + 2) - a(a - 3) - 4$

1.15 The reliability of a system with 3 units in parallel is given by $1 - (1-R_1)(1 - R_2)(1 -R_3)$ where R_1, R_2, R_3 are the individual reliabilities of the three parallel units. Show that this expression is equal to

$$R_1+R_2+R_3-R_1R_2-R_1R_3-R_2R_3+R_1R_2R_3$$

1.16 A body moving under constant acceleration a with initial velocity u has velocity v after time t given by $v = u + at$. The distance travelled by the body is given by

$$s = \frac{(u+v)t}{2}$$

Substitute for v in the expression for s to show that s can be found in terms of u, a and t by the formula

$$s = ut + \frac{1}{2}at^2$$

Chapter 2 Fractions and Fractional Expressions

2.1 INTRODUCTION

As mentioned in Section 1.6, we need to understand the rules governing division because of the need to have fractional expressions. In this text we shall not attempt to go through all the rules of fractions by looking at cakes divided into parts. This is an excellent way of checking that rules involving fractions do make sense, but we assume that you have developed sufficient graphical skills to make up such examples for yourself, and we are aiming here to give a summary of the results and how they are used.

A fraction, or fractional expression, has names associated with the top and bottom lines.

$$\begin{array}{c} \textit{numerator} \\ \downarrow \\ \dfrac{a}{b} \\ \uparrow \\ \textit{denominator} \end{array}$$

2.2 FRACTIONAL NUMBERS

If a number is expressed as a fraction, in general the best approach is to convert it to a decimal by performing the division on a calculator. This will usually give an approximate value for the number, as only fractions which, in their lowest form, have denominators with factors 2 and 5 can be expressed exactly in decimal notation. Decimal numbers are easy to multiply and add etc. Depending on the accuracy of your calculator, however, you can end up with some interesting results by always using decimal form to perform a calculation. Try the following:

$$\frac{3}{9} \times 9$$

You could well get the result 2.999999 and then decide to round that to 3. Of course in this case a quick look at the problem should tell you that the answer is 3. This is an example of a simple problem that is better done without a calculator by using the rules of fractions.

In more complicated calculations it is important to watch out for the order of performing the operations:

$$\frac{3\frac{4}{9}+4}{-6-25\times\frac{3}{7}}$$

Notice that $3\frac{4}{9}$ means $3+\frac{4}{9}$. The easiest way to find the value of the entire expression is to calculate the bottom line first, store this in memory and then calculate the top line. Finally divide the top by the bottom line. Do this yourself - you should get -1.0527 to four decimal places.

For the rest of this chapter we will concern ourselves with the rules of manipulating fractions, mainly in order to be able to simplify fractional expressions.

2.3 FRACTIONS WITH DENOMINATOR 1

Divide anything by 1 and it will not be changed. To go back to our modelling ideas introduced in Chapter 1, we could convince ourselves of this by thinking of a problem such as 'There were three cream cakes on the trolley and only one person left in the dining room'. Such a situation would result in the poor unfortunate person eating all three cakes, hence $\frac{3}{1} = 3$. In general, this leads to $\frac{a}{1} = a$.

This rule can be very useful to get rid of denominators of 1, or it can be used to write any simple expression as a fractional one, should we so wish.

2.4 MULTIPLYING FRACTIONS

To multiply fractions, multiply the numerators together and the denominators together.

$$\frac{a}{b}\cdot\frac{c}{d} = \frac{ac}{bd}$$

By using the result of Section 2.3 we can now multiply any two expressions together, whether fractional or not.

$$\frac{x+1}{x}\cdot(x-5) = \frac{x+1}{x}\cdot\frac{(x-5)}{1} = \frac{(x+1)(x-5)}{x}$$

Here we have written (x-5) as a fraction with denominator 1, and then performed the multiplication.

We can also rewrite expressions such as $\frac{3}{4}a$ as $\frac{3a}{4}$ by using this same argument.

2.5 EQUIVALENT FRACTIONS

From the identity law any number multiplied by 1 remains unchanged:

$a.1 = a$

Also, as any number divided by itself gives 1, there are many ways of writing the number 1:

$$1 = \frac{1}{1} = \frac{2}{2} = \frac{3}{3} = \frac{4}{4} = \frac{5}{5} = \ldots = \frac{1000}{1000} = \frac{8.2}{8.2}$$

Any fraction will be unchanged in value if both the top and bottom lines are multiplied by the same number, because we are only multiplying by 1, e.g.

$$\frac{3}{4} = \frac{3 \times 2}{4 \times 2} = \frac{3 \times 3}{4 \times 3} = \frac{3 \times 10}{4 \times 10}$$

hence

$$\frac{3}{4} = \frac{6}{8} = \frac{9}{12} = \frac{30}{40}$$

This can be extended to say that a fractional expression is not changed by multiplying the top and bottom lines by the same thing (in general, however, be careful to avoid something that could equal 0).

Considering a/b where b $\neq$ 0 then

$$\frac{a}{b} = \frac{a.a}{b.a} \qquad \text{if a} \neq 0$$

and

$$\frac{a}{b} = \frac{a(b-1)}{b(b-1)} \quad \text{if b} \neq 1$$

In the same way, a fractional expression is not changed by cancelling the same expression from the top and bottom lines. Be careful though: whatever is cancelled must be a factor of all of the numerator and of the denominator. For example:

$$\frac{15}{24} = \frac{5}{8}$$

Here the factor 3 was cancelled from the top and bottom lines. Also

$$\frac{3a + 3b}{3c} = \frac{3(a + b)}{3c} = \frac{a + b}{c}$$

where we have cancelled the common factor of 3 and

$$\frac{(x - 1)x}{x(x - 2)} = \frac{x - 1}{x - 2} \qquad \text{where } x \neq 0$$

where we have cancelled the common factor of x. This last equality is only true if x ≠ 0, because the left-hand side has no defined value for x = 0.

Be careful when cancelling to ensure that you cancel correctly. For instance note the following:

$$\frac{a + 3b}{3a} \neq \frac{a + b}{a}$$

3 is not a factor of the whole of the top line of the fraction so no cancellation can be performed in this case.

2.6 THE DISTRIBUTIVE LAW, OR SPLITTING THE LINE

As established in Section 1.6

$$\frac{a + b}{c} = \frac{a}{c} + \frac{b}{c}$$

This is the rule which allows fractions with a sum on the top line to be split into the sum of two fractions:

$$\frac{1 + 2}{3} = \frac{1}{3} + \frac{2}{3}$$

Similarly, it gives the rule for summing two fractions with the same denominator:

$$\frac{1}{3} + \frac{2}{3} = \frac{1 + 2}{3}$$

The following example, where x ≠ 0, uses the rule in both ways:

$$\frac{x + 1}{x} + \frac{x + 2}{x} = \frac{x + 1 + x - 2}{x}$$

$$= \frac{2x-1}{x}$$

$$= \frac{2x}{x} - \frac{1}{x}$$

$$= 2 - \frac{1}{x}$$

Be careful, because the horizontal line acts in the same way as a bracket, which can lead to complications with minus signs. An example of this is

$$-\frac{a+b}{c} = \frac{-(a+b)}{c} = \frac{-a-b}{c} = -\frac{a}{c} - \frac{b}{c}$$

Always remember that the bottom line cannot be split in the same way as the top line i.e.

$$\frac{a}{b+c} \neq \frac{a}{b} + \frac{a}{c}$$

2.7 ADDING FRACTIONS

The method of the previous section allows us to add fractions with a common denominator, but we need a rule to be able to sum any two fractions or fractional expressions. This is done by 'finding the common denominator', e.g

$$\frac{3}{4} + \frac{5}{6}$$

The smallest number that both 4 and 6 will go into is 12 (the lowest common denominator). In practice it may be easier to simply multiply the denominators together (giving 24) and use that for the common denominator.

Now write the two fractions in terms of the common denominator. We solve the two puzzles. First,

$$\frac{3}{4} = \frac{?}{12}$$

Here the question mark must be 9, as we have multiplied the top and bottom line by 3.

$$\frac{5}{6} = \frac{?}{12}$$

Here the question mark must be 10 as we have multiplied the top and bottom line by 2.

We then get

$$\frac{3}{4} + \frac{5}{9} = \frac{9}{12} + \frac{10}{12} = \frac{9+10}{12} = \frac{19}{12}$$

As the fractions now have a common denominator they have been combined to give a sum of

$$\frac{19}{12} = \frac{12}{12} + \frac{7}{12} = 1\frac{7}{12}$$

Combining two fractional expressions is very similar. For example:

$$\frac{x-1}{x+2} + \frac{x}{x-3} \qquad \text{where } x \neq \text{-2 and } x \neq 3$$

The common denominator is (x+2)(x-3) and we rewrite both fractions:

$$\frac{x-1}{x+2} = \frac{(x-1)(x-3)}{(x+2)(x-3)} \qquad \text{(multiplying the top and bottom by (x-3))}$$

and

$$\frac{x}{x-3} = \frac{x(x+2)}{(x-3)(x+2)} \qquad \text{(multiplying the top and bottom by (x+2))}$$

Hence

$$\frac{x-1}{x+2} + \frac{x}{x-3} = \frac{(x-1)(x-3) + x(x+2)}{(x+2)(x-3)}$$

2.8 DIVIDING FRACTIONS

Again we use the idea of equivalent fractions to perform division of fractions. For example, we wish to write

$$\frac{\frac{2}{3}}{\frac{3}{5}}$$

as a single fraction. Before changing this expression, it is important to understand the order of operations in this expression. The longer line means that there are implied brackets on the top and bottom of that line, i.e.

$$\frac{\dfrac{2}{3}}{\dfrac{3}{5}}$$

means

$$\frac{\left(\dfrac{2}{3}\right)}{\left(\dfrac{3}{5}\right)}$$

To make this into one simple fraction we get rid of the denominator by multiplying by the multiplicative inverse of 3/5 i.e. the reciprocal, 5/3. To do this without changing the value of the expression we must multiply both the numerator and the denominator, hence using the idea of equivalent fractions.

$$\frac{\dfrac{2}{3}}{\dfrac{3}{5}} = \frac{\dfrac{2}{3}\dfrac{5}{3}}{\dfrac{3}{5}\dfrac{5}{3}} = \frac{\dfrac{10}{9}}{1} = \frac{10}{9}$$

The same approach will work for a fractional expression.

To simplify

$$\frac{\dfrac{3}{y-3}}{\dfrac{y}{y+1}}$$

multiply the numerator and denominator by (y + 1)/y, which gives

$$\frac{\dfrac{3}{y-3}}{\dfrac{y}{y+1}} = \frac{\dfrac{3}{y-3}\dfrac{y+1}{y}}{\dfrac{y}{y+1}\dfrac{y+1}{y}} = \frac{\dfrac{3(y+1)}{y(y-3)}}{1}$$

$$= \frac{3(y+1)}{y(y-3)}$$

2.9 SUMMARY

1. To deal with a complicated numerical problem involving fractions, convert to decimal form by using a calculator. This will, in general, give only an approximate result.
2. It is necessary to know the rules of fractions in order to simplify fractional expressions.
3. To multiply two fractions multiply the top lines (numerators) and the bottom lines (denominators).

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$$

4. Any simple expression can be written as a fraction with denominator 1:

$$a = \frac{a}{1}$$

5. A fraction will not change its value if multiplied by the same expression on the top and bottom lines because this is the same as multiplying by 1. However, zero must be avoided.

$$\frac{a}{b} = \frac{ax}{bx} \qquad\qquad x \neq 0$$

6. The distributive law for division means that a fraction with a sum in the numerator can be split into the sum of two fractions.

$$\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$$

7. This cannot be done with the denominator

$$\frac{a}{b+c} \neq \frac{a}{b} + \frac{a}{c}$$

8. The sum of two fractions can be expressed as a single fraction by writing them over a common denominator:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$$

9. To divide fractions, multiply the top and bottom lines by the reciprocal of the bottom line. By Point 5 above, this does not change the value of the expression.

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{\frac{a}{b}\frac{d}{c}}{\frac{c}{d}\frac{d}{c}} = \frac{\frac{ad}{bc}}{1} = \frac{ad}{bc}$$

2.10 **EXERCISES**

2.1 Calculate the following

(a) $\quad 4\frac{27}{9}+\frac{10}{3}\times 9$

(b) $12.6 \; - \; \dfrac{4-\dfrac{5}{8}}{\dfrac{3}{7}+2\dfrac{12}{17}}$

2.2 Simplify

$$\frac{\frac{a}{1}+\frac{1}{b}}{1}+a$$

2.3 Simplify

$$5\cdot\frac{2b}{a+2} \; \cdot \; (a+3)$$

2.4 Write $\dfrac{ac}{bd}$ as the product of two terms, in as many different ways as you can think of.

2.5 Simplify, specifying those values for which the expression is not defined.

(a) $\dfrac{x+xy}{x(x+1)}$

(b) $\dfrac{a+ba+ac}{ad+ae+af}$

2.6 Simplify

$$-\frac{3x-2}{x(x-1)}+\frac{1}{x(x-1)} \qquad \text{where x} \neq 0 \text{ and x} \neq 1$$

2.7 Express as a single fraction

(a) $\dfrac{x}{x-1}-\dfrac{x-1}{x}$

(b) $\dfrac{a}{(a-b)(a-c)}+\dfrac{b}{(a-b)(a-d)}+\dfrac{c}{(a-b)}$

2.8 Simplify, where x $\neq 0.5$ and x $\neq 0$

$$\frac{\left(\dfrac{x+1}{2x-1}\right)}{\left(\dfrac{4}{2x}\right)}$$

2.9 Find the following:

(a) $4\dfrac{2}{5}+\dfrac{3}{5}-\dfrac{4}{6}+\dfrac{3}{6}$

(b) $4-\dfrac{1}{4}+2\dfrac{5}{8}-\dfrac{11}{7}$

(c) $\dfrac{4-2}{8}+\dfrac{-6}{-3}-\dfrac{18}{3}\left(\dfrac{1}{3}-\dfrac{7}{16}\right)$

2.10 Express the following as single fractions:

(a) $\dfrac{2}{y+1}-\dfrac{1}{y+3}$

(b) $\dfrac{3}{y-8}-\dfrac{2}{8-y}$

(c) $\dfrac{y-1}{y(y+1)}-\dfrac{y}{(y+1)(y+2)}$

(d) $\dfrac{2}{3y} - \dfrac{3}{2y}$

(e) $\dfrac{y+2}{y+3} - \dfrac{y+1}{y+2}$

2.11 Simplify the following:

(a) $\dfrac{\dfrac{3}{8}}{\dfrac{9}{6}}$

(b) $\dfrac{\dfrac{1}{3}}{y}$

(c) $\dfrac{\dfrac{y+1}{y-2}}{\dfrac{y}{y-2}}$

(d) $\dfrac{\dfrac{y-3}{y+1}}{\dfrac{1}{y} + \dfrac{2}{y+1}}$

(e) $\dfrac{\dfrac{3}{y} - \dfrac{2}{y^2}}{y+1}$

2.12 State whether the following are true or false and justify your answer:

(a) $\dfrac{1}{x-2} = \dfrac{1}{x} - \dfrac{1}{2}$ where x $\neq$ 2 and x $\neq$ 0

(b) $-\dfrac{x-2}{x} = \dfrac{2}{x} - 1$ where x $\neq$ 0

(c) $\dfrac{x+3}{(x+1)(x+3)} = 1 + \dfrac{1}{x+1}$ where x $\neq$ -1 and x $\neq$ -3

(d) $\dfrac{1}{x+1} - \dfrac{1}{x-1} = \dfrac{-2}{x^2-1}$ where x ≠ -1 and x ≠ 1

2.13 The inductance, $L\,\mu$ H, of a circular cross-sectional coil can be estimated by the formula:

$$L = \frac{0.0787N^2d^2}{3.5d + 8l}$$

where d = diameter of the coil (cm), l = length of winding (cm), N = number of turns. If d = $1\frac{2}{3}$ cm, l = $3\frac{3}{4}$ cm and N = 220, calculate the inductance L.

2.14 The circumferential strain on a compressed air cylinder is given by

$$\varepsilon_c = \frac{\sigma_c}{E} + v\frac{\sigma_\gamma}{E} - v\frac{\sigma_l}{E}$$

where σ_c, σ_l, σ_r are the circumferential, longitudinal, and radial stresses respectively and v, E are Poisson's ratio and the modulus of elasticity for the material. If v = 0.28, σ_c = 78MNm^{-2}, σ_l = 38 M N m^{-2}, σ_γ = 26MNm^{-2}, E = 198 G N m^{-2} calculate ε_c.

Chapter 3 Equations

3.1 INTRODUCTION

An equation is a statement e.g

$$x + 2 = 5$$

which may or may not be true, depending on the value which is substituted for the 'unknown', in this case x. In the above example, if we substitute $x = 3$ then the equation becomes

$$3 + 2 = 5$$

which is true. However, if we substitute $x = 4$ we get

$$4 + 2 = 5$$

which is false. The value(s) which make(s) the equation into a true statement are called the solution(s). To solve an equation means to find all the solutions. Not all equations can be solved by the use of algebraic manipulation. Only in a few simple cases is this possible. Most solution methods involve using a numerical method and therefore are generally performed using a computer. It is, however, useful to know how to solve equations in those simple cases where it can be done by algebraic manipulation. Another thing to watch out for is that not all equations have solutions. A quadratic equation like $x^2 = -4$ has no real solutions. One way to solve an equation is to take a guess at the solution and substitute in the guess to see if it is correct. However, this can be a long process and may run into difficulties if we don't know how many solutions there are. We therefore present some methods for solving simple equations.

3.2 INVERSE OPERATIONS

One way to solve a simple equation is to think of an equation as a puzzle. There are two people A and B, and A proposes the problem:

$$2x+3 = 9$$

by saying, 'I have thought of a number, multiplied it by 2 and then I added 3, the result is 9. What is the number I first thought of?'

B immediately realises that the original number can be found by going through the whole process backwards. Hence B thinks, 'I start with 9, subtract 3 (giving 6) and divide by 2 (giving 3). Hence the original number, x, is 3'.

This method uses the idea of inverse operations to solve the equation. The equation can be viewed as a flow diagram with input x and boxes indicating the operations. The inverse flow diagram is then found by reversing the order and replacing the original operations by their inverses. A flow diagram for $2x + 3 = 9$ and the inverse flow diagram giving its solution is shown in Fig. 3.1. This method of using flow diagrams illustrates the use of inverse operations in the process of solving simple equations.

Equations are more usually solved in stages. At each stage a new equation is found which is equivalent to the original equation.
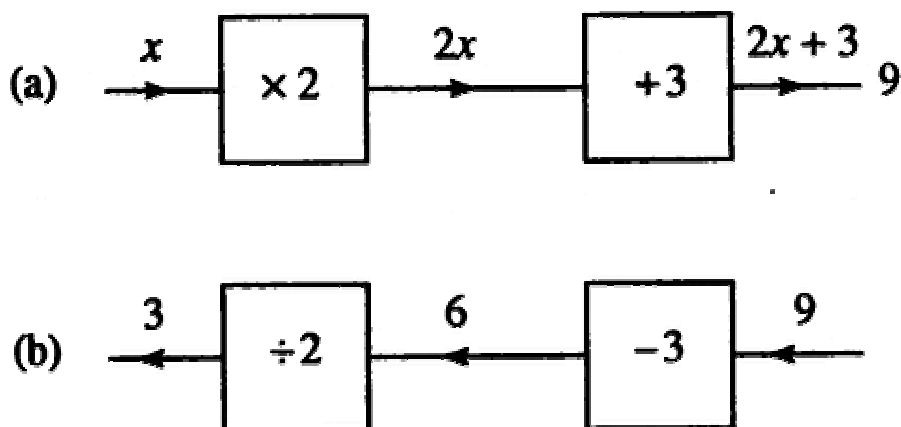

Figure 3.1 (a) A flow diagram describing the equation $2x + 3 = 9$ and (b) the inverse diagram, giving the solution as $x = (9 - 3)/2$

3.3 EQUIVALENT EQUATIONS

An equation can be viewed as a pair of scales. The scales start off in an equilibrium position with the left-hand weight exactly balancing the right-hand weight. In order to maintain the balance then we can add weights, subtract weights or multiply or divide by some value but only if we do exactly the same to both sides. If we perform the same operation on both sides of the equals sign and we maintain the balance of the equation we say that the new equation is equivalent to the original equation. Equivalence can be indicated by using the symbol $\Leftrightarrow$ or $\equiv$.

Example 3.1 $2x+3 = 9$ can be viewed as the situation where two unknown weights are on one side of the scales with three other one unit weights. On the other side are nine one-unit weights. This is pictured in Fig. 3.2.

To find x, first remove 3 one unit weights from both sides as in Fig 3.3, showing that $2x + 3 = 9 \Leftrightarrow 2x = 6$. Now divide both sides by 2, giving Fig. 3.4, showing that $2x = 6 \Leftrightarrow x = 3$.
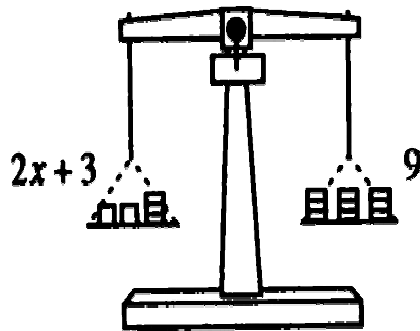
Here we have found the solution, $x = 3$.

Figure 3.2. The equation 2x+3 = 9 can viewed as the situation where two unknown weights are on one side of the scales with three other one unit weights. On the other side are nine one-unit weights and the scales are balanced.
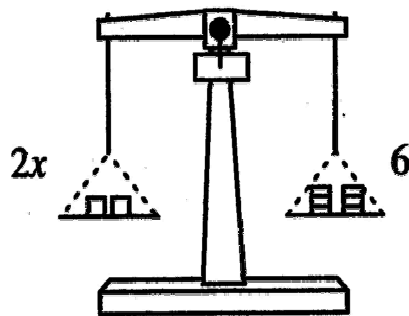

Figure 3.3. The scales pictured in Figure 3.2 have had three one unit weights removed from both sides, so they still balance and represent the equation 2x = 6. This shows that 2x+3 = 9 $\Leftrightarrow$ 2x = 6.
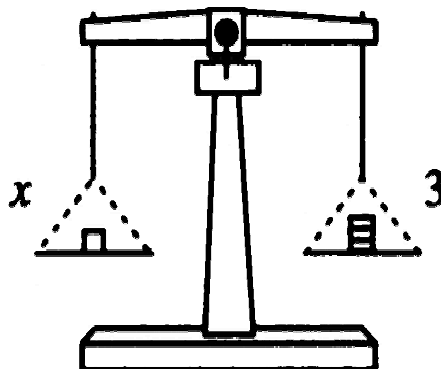

Figure 3.4. The scales pictured in Figure 3.3 have had the weights in both the scale pans halved so they still balance and represent the equation $x = 3$. This shows that 2x = 6 $\Leftrightarrow$ $x = 3$.

We solved the equation in a number of steps, ensuring that at each stage we have an equation equivalent to the original equation. Equivalent equations were found by 'doing the same thing to both sides'. To decide *what* to do to both sides we use the idea of inverse operations. The aim is to reach the position when only the unknown quantity remains on one side of the equation. For convenience we shall call the unknown quantity x. The coefficient of x is the number multiplying x.

How to solve a simple equation

1. Remove any brackets where necessary.
2. 'Collect' everything involving x onto one side of the equation and anything not involving x onto the other side. (This is done by a number of additions and subtractions to both sides of the equation).
3. Combine all the terms involving x so there is a single coefficient of x.
4. Divide both sides of the equation by the coefficient of x.
5. Check that the answer is correct by substituting into the original equation to see whether the statement is true with this value of x.

Example 3.2 Solve the equation $2(3 - 3x) + 5x = -3x$

SOLUTION

1. *Remove brackets* $\qquad\qquad\qquad 6 - 6x + 5x = -3x$

$$\Leftrightarrow 6 - x = -3x$$

2. *Collect terms*

Collect the x terms onto the left-hand side of the equation (by adding $3x$ to both sides of the equation) and all other terms onto the right-hand side of the equation (by subtracting 6 from both sides of the equation). Here we write out the intermediate stages

$6 - x + 3x = -3x + 3x$ $\qquad$ (adding 3x to both sides)

$\Leftrightarrow 6 + 2x = 0$ $\qquad\qquad$ (combining terms)

$\Leftrightarrow 6 + 2x - 6 = 0 - 6$ $\qquad$ (subtracting 6 from both sides)

$\Leftrightarrow 2x = -6$

3. *Divide by the coefficient of x*

$\dfrac{2x}{2} = \dfrac{-6}{2}$ $\qquad\qquad$ (Divide both sides by 2)

$$\Leftrightarrow x = -3$$

4. *Check answer*

Substitute $x = -3$ into the original equation:

$$2(3 - 3x) + 5x = -3x$$

giving
$$2(3 - 3(-3)) + 5(-3) = -3(-3)$$

$$\Leftrightarrow 2(3 + 9) - 15 = 9$$

$$\Leftrightarrow 24 - 15 = 9$$

which is true. Therefore $x = -3$ is the solution to the equation

$$2(3 - 3x) + 5x = -3x$$

The equations we have met so far are called linear equations because they only involve terms in x and constant values. Linear equations have exactly one solution. Solving quadratic equations is not so straightforward.

3.4 SOLVING QUADRATIC EQUATIONS

Quadratic equations are those that involve x^2 and possibly x but no higher powers of x. Quadratic equations cannot always be solved by using inverse operations as described so far. To understand why this is so, try thinking of the puzzle approach which we used in Section 3.3. Person A proposes the puzzle:

$$x^2 - 2x = 3$$

by saying, 'I thought of a number, multiplied it by itself then subtracted twice the original number, this gave me the answer 3. What is the number I first thought of?'

Person B now tries to go through the puzzle backwards by saying, 'Start with 3, add on twice the number that A first thought of.…', here B is stuck because it is not possible to add on twice a value which has not yet been discovered. This is pictured as a flow diagram in Fig. 3.5.
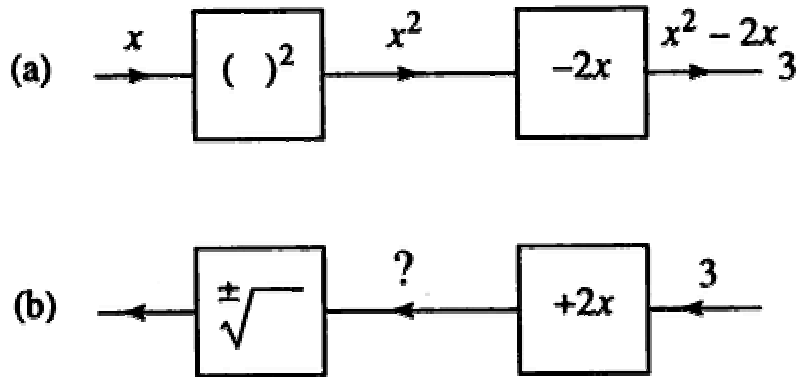
Figure 3.5 An attempt to solve the equation $x^2 - 2x = 3$ by using a flow diagram. (a) The flow diagram for $x^2 - 2x = 3$. (b) The inverse flow diagram cannot be completed the output from the '+ 2x' operation requires a knowledge of the still unknown quantity x

Supposing A, taking pity on B, proposed a simpler problem.

$$x^2 - 4 = 5$$

Now A says, 'I have thought of a number, multiplied it by itself, subtracted 4 and the result is 5'.

B, not wishing to appear disheartened, has another go at going through the problem backwards: 'Start with 5, add on 4 (giving 9), now take the square root of 9, giving +3 or –3'. Notice here that B knows that the 'inverse' of squaring a number is taking the square root. It is not an exact inverse because both 3^2 and $(-3)^2$ give 9, so there are two possibilities when taking the square root of 9.

Jubilantly B announces to A that the original number was +3 or –3. Of course A could be awkward and insist that B tells her exactly what was her original number and that a choice of two numbers is not good enough. In such circumstances B would lose because it is impossible to say exactly the number that A started with. However, let's assume that A accepts the idea of two solutions to the puzzle. Why then was it possible to solve $x^2 - 4 = 5$ by this method, when it was not possible to solve $x^2 - 2x = 3$ by the same method? The problem occurs because of the term in x.

Completing the square

There is a method called 'completing the square' which gets round the problem of solving a quadratic equation by writing all the x^2 terms and the x terms into an exact square. Just for interest we will look at $x^2 - 2x = 3$ using this method. The left-hand side of the equation can be rewritten using a complete square and a constant value:

$$x^2 - 2x = (x-1)^2 - 1$$

This last expression is true, whatever the value of x. Check this by multiplying out the brackets. Now we can rewrite the original problem as

$$(x–1)^2–1 = 3$$

Now A's original puzzle has been reworded to 'I thought of a number and subtracted 1, then squared the result and subtracted 1. Finally I had the number 3. What number did I first think of?'

B can now go through the problem backwards. 'Start with 3, add on 1 (giving 4), take the square root (giving +2 or –2) then add on 1, giving +3 or –1". Hence B has solved the puzzle.

This process can be shown in a flow diagram as in Fig. 3.6.
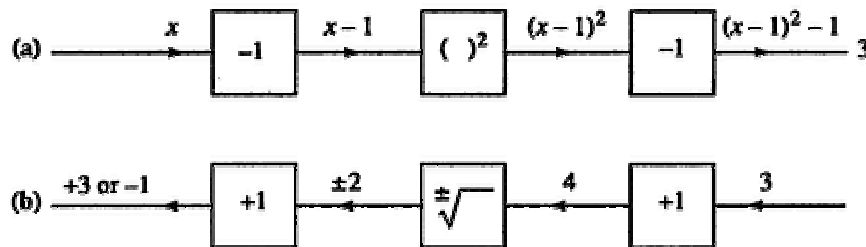
Figure 3.6 The equation $x^2–2x = 3$ has been rewritten as $(x–1)^2–1 = 3$ by 'completing the square'. The inverse flow diagram of this equation can be completed. (a) The flow diagram for $(x–1)^2–1 = 3$. (b) The inverse flow diagram giving the solution as $\pm\sqrt{3+1}+1$.

Rather than go through this process for every single quadratic equation, the square has been completed for the general quadratic equation $ax^2 + bx + c = 0$, giving the formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Note that no real number squared is negative, and hence the square root in this expression cannot be found if $b^2 - 4ac$ is negative. There are no real solutions to a quadratic equation in this case. Also, if $b^2 - 4ac$ is zero then there is only one unique solution to the quadratic equation instead of two.

This then gives our first method for solving quadratic equations:

1. Take out all unnecessary brackets and collect all the terms on one side of the equation, leaving zero on the other side
2. Use the formula if $ax^2 + bx + c = 0$ then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

3. Substitute any solutions found into the original equation to check they are correct.

Example 3.3 Solve $x(x - 2) = 3$.

1. *Remove brackets and collect terms*

$x(x-2) = 3 \Leftrightarrow x^2 - 2x = 3 \Leftrightarrow x^2 - 2x - 3 = 0$

2. *Use the formula*

$a = 1$, $b = -2$, $c = -3$ giving

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-3)}}{2}$$

$$\Leftrightarrow x = \frac{2 \pm \sqrt{16}}{2}$$

$$\Leftrightarrow x = \frac{2 + 4}{2} \text{ or } x = \frac{2 - 4}{2}$$

$$\Leftrightarrow x = 3 \text{ or } x = -1$$

3. *Check the solutions*

Substitute $x = 3$ into the original equation $x(x - 2) = 3$, giving $(3)(3 - 2) = 3$, which is true. Substitute $x = -1$ into $x(x - 2) = 3$, giving $(-1)(-1 - 2) = 3$ which is true Both solutions are correct so $x = 3$ or $x = -1$ are solutions to the equation $x(x - 2) = 3$.

The second method of solving quadratic equations uses a completely different approach and is based on the fact that if two numbers or expressions multiply together to give 0 then one or the other must be 0. This method is only used for simple quadratic equations where it is easy to spot a factorization.

1. Take out all unnecessary brackets and collect all the terms onto one side of the equation, leaving 0 on the other side.
2. Factorize into two brackets, each one containing an x term.
3. Use the fact that if two things multiplied together give 0 then one or the other must be 0 to reduce the equation to two linear equations.
4. Solve the two linear equations.

    5. Substitute both solutions into the original equation to check they are correct.

Example 3.4 Solve $x(x - 2) = 3$.

1. *Remove brackets and collect terms*

$x(x - 2) = 3 \Leftrightarrow x^2 - 2x = 3 \Leftrightarrow x^2 - 2x - 3 = 0$

2. *Factorize*

$x^2 - 2x - 3 = 0 \Leftrightarrow (x-3)(x+1) = 0$

3. *Reduce to 2 linear equations*

$(x-3)(x+1) = 0 \Leftrightarrow x-3 = 0$ or $x+1=0$

4. *Solve the linear equations*

$x-3 = 0$ or $x+1 = 0 \Leftrightarrow x = 3$ or $x = -1$

5. *Check the solutions*

Substitute $x = 3$ into the original equation $x(x-2) = 3$ giving $(3)(3-2) = 3$ which is true.

Substitute $x = -1$ into $x(x-2) = 3$ giving $(-1)(-1-2) = 3$ which is true.

Both solutions are correct so $x = 3$ or $x = -1$ are solutions to the equation $x(x-2) = 3$.

3.5 SOLVING EQUATIONS INVOLVING FRACTIONAL EXPRESSIONS

Some equations with fractional expressions will transform to linear or quadratic equations. To solve such equations multiply the whole equation by a common denominator. The equation can then be solved in the ways we have already met.

Example 3.5 Solve the following for x:

$$\frac{1}{3} = \frac{5}{x-1} + \frac{6}{x-2}$$

SOLUTION

$$\frac{1}{3} = \frac{5}{x-1} + \frac{6}{x-2}$$

Multiply by 3(*x*-1)(*x*-2) Note that because the equation involves fractional expressions involving *x*, then values which make the denominators 0 would give a division by 0, which is undefined. Such values must be excluded from the possible solutions.

$$\frac{1}{3} = \frac{5}{x-1} + \frac{6}{x-2}$$

$$\Leftrightarrow 3(x-1)(x-2)\tfrac{1}{3} = 3(x-1)(x-2)\left(\frac{5}{x-1} + \frac{6}{x-2}\right) \quad \text{where } x \neq 1 \text{ and } x \neq 2$$

$$\Leftrightarrow (x-1)(x-2) = 15(x-2) + 18(x-1)$$

$$\Leftrightarrow x^2 - x - 2x + 2 = 15x - 30 + 18x - 18$$

$$\Leftrightarrow x^2 - 3x + 2 = 33x - 48$$

$$\Leftrightarrow x^2 - 3x - 33x + 2 + 48 = 0$$

$$\Leftrightarrow x^2 - 36x + 50 = 0$$

$$\Leftrightarrow x = \frac{36 \pm \sqrt{1296 - 200}}{2}$$

$$\Leftrightarrow x = \frac{36 \pm \sqrt{1096}}{2}$$

$$\Rightarrow x \approx \frac{36 - 33.106}{2} \quad \text{or} \quad x \approx \frac{36 - 33.106}{2} \quad \text{(to 3 d.p.)}$$

$$\Leftrightarrow x \approx 34.553 \quad \text{or} \quad x \approx 1.447 \quad \text{(to 3 d.p.)}$$

Check

Substitute *x* = 34.553 into the original equation:

$$\frac{1}{3} = \frac{5}{x-1} + \frac{6}{x-2}$$

$$\frac{1}{3} = \frac{5}{34.553 - 1} + \frac{6}{34.553 - 2}$$

which gives 0.333 = 0.149 + 0.184, which is true.

Substitute *x* =1.447 into the original equation

$$\frac{1}{3} = \frac{5}{1.447 - 1} + \frac{6}{1.447 - 2}$$

giving 0.333 = 11.1857 - 10.8499, which is correct to one decimal place.

3.6 REARRANGING FORMULAE

The same ideas we have used to solve equations can be used to rearrange formulae.

Example 3.6 Two resistors, of resistances R_1 and R_2, arranged in an electrical circuit in parallel, have equivalent resistance R where

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

Express R_1 as the subject of the formula.

SOLUTION. Multiply by the common denominator $R\,R_1\,R_2$:

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

$$\Leftrightarrow RR_1R_2\left(\frac{1}{R}\right) = RR_1R_2\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

where R, R_1, R_2 are non zero

$$\Leftrightarrow R_1\,R_2 = R_1\,R_2 + R_1\,R_2$$

Collect all the terms involving R_1 on one side of the equation and others terms on the other side of the equation:

$$R_1\,R_2 = R\,R_2 + R\,R_1 \Leftrightarrow R_1\,R_2 - R\,R_1 = R\,R_2$$

Factorize, taking out of the common bracket the common factor of R_1:

$$\Leftrightarrow R_1\,(R_2 - R) = RR_2$$

Now divide by the term multiplying R_1 ie $(R_2 - R)$

$$\frac{R_1(R_2 - R)}{(R_2 - R)} = \frac{RR_2}{(R_2 - R)} \qquad\qquad \text{where } (R_2 - R) \neq 0 \text{ (i.e. } R_2 \neq R)$$
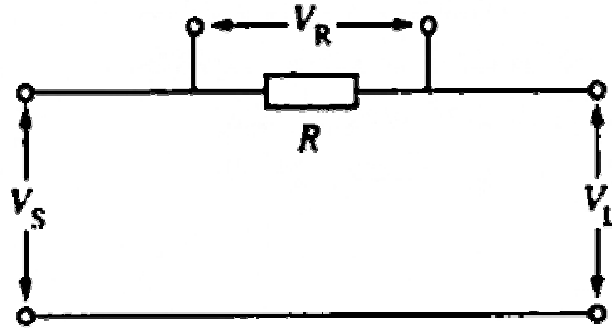
$$R_1 = \frac{RR_2}{R_2 - R}$$

Figure 3.7 The circuit for example 3.7

Example 3.7 In the high frequency a.c. circuit described in Fig. 3.7, the power into a load, P, can be expressed in terms of the voltage of the supply, V_s, the voltage across the load V_L, the voltage across the resistor V_R and the resistance R, by the expression

$$P = \frac{V_S^2 - (V_L^2 + V_R^2)}{2R}$$

Express V_L as the subject of the formula.

SOLUTION.

$$P = \frac{V_S^2 - (V_L^2 + V_R^2)}{2R}$$

$$\Leftrightarrow 2RP = V_S^2 - (V_L^2 + V_R^2) \qquad \text{(multiplying both sides by } 2R\text{)}$$

$$\Leftrightarrow 2RP = V_S^2 - V_L^2 - V_R^2 \qquad \text{(removing the bracket)}$$

$$\Leftrightarrow V_L^2 = V_S^2 - V_R^2 - 2RP$$

(putting the term involving V_L on one side of the equation and all other terms on the other side by adding V_L^2 to both sides and subtracting $2RP$ from both sides)

$$\Leftrightarrow V_L = \pm\sqrt{V_S^2 - V_R^2 - 2RP}$$

45

(taking the square root of both sides)

As V_L must be positive the negative result can be discarded, giving:

$$V_L = \sqrt{V_S^2 - V_R^2 - 2RP}$$

3.7 SUMMARY

1. An equation is a statement, involving some unknown, x, which may be true or false.
2. A solution is a value for x, which makes the equation into a true statement.
3. Equations are solved if we know all the solutions.
4. Equations remain equivalent if the same operation is performed to both sides of the equation. The symbols for equivalence are $\Leftrightarrow$ or $\equiv$.
5. Linear equations can be solved using inverse operations and by doing the same thing to both sides.
6. Quadratic equations, $ax^2 + bx + c = 0$, can be solved by using the formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

or by using factorization.

7. Many equations involving fractional expressions can be transformed to a linear or quadratic equation by multiplying by a common denominator.
8. Formulae may be rearranged to change the subject of the formula using the same techniques used for solving equations.

3.8 EXERCISES

3.1 Solve the following equations:

(a) $12 = 6-3x$

(b) $y = 19-3y$

(c) $4(z-2) = 12$

(d) $5(a+3) = 25$

(e) $\frac{x}{9} = 18$

(f) $\frac{28}{a} = 7$

g) $4p - 2 = 3p + 3$

(h) $\frac{x}{6} + \frac{x-2}{5} = 4$

(i) $\frac{p}{2} + \frac{p+4}{3} = 18$

(j) $\dfrac{x-6}{9} = \dfrac{3+x}{6}$ (k) $\dfrac{3x+3}{5} - \dfrac{2x-3}{15} = -16$

(l) $\dfrac{p}{p+1} = \dfrac{1}{p} + 1$

(m) $\dfrac{y-1}{y} = \dfrac{y}{y-3}$ (n) $\dfrac{x}{x+2} - \dfrac{1}{x-2} = 1$

3.2 Solve the following equations:

(a) $y^2 + 17y = 60$ (b) $z^2 - 3z = 10$ (c) $p^2 = 8p - 15$

(d) $(x-2)(x+1) = 18$ (e) $\dfrac{y}{y-3} = y - 4$ (f) $\dfrac{6}{y} = y + 5$

(g) $z + \dfrac{12}{z} = 7$ (h) $4x^2 + 10x + 3 = 0$ (i) $y^2 - 5y + 2 = 0$

(j) $z^2 + 3z - 5 = 0$ (k) $q^2 - q - 3 = 0$ (l) $x + \dfrac{1}{x} = 3$

m) $(y + 1)(y + 2) = 28$

3.3 The lens maker formula states that $\dfrac{1}{v} + \dfrac{1}{u} = \dfrac{1}{f}$ where v is the distance of the object from the lens, u is the distance of the image from the lens and f is the focal length. Write u as the subject of the formula.

3.4 An object moving at constant acceleration, a, moves a distance s in time t when its initial velocity is u. s is given by $s = ut + \frac{1}{2}at^2$. Make t the subject of the formula and, assuming that t is positive find t when

(a) $s = 100\text{m}$, $u = 1\text{m s}^{-1}$, $a = 3\text{m s}^{-2}$

(b) $s = 205\text{m}$, $u = 10\text{m s}^{-1}$, $a = 1\text{m s}^{-2}$

3.5 A laterally insulated metal bar, of constant cross sectional area A and of length x, is maintained at a temperature T_1 at one end and at a temperature T_2 at the other end (see Fig. 3.8). The heat Q crossing any cross-section of the bar in time t is given by

$$\frac{Q}{t} = \lambda A\left(\frac{T_2 - T_1}{x}\right)$$

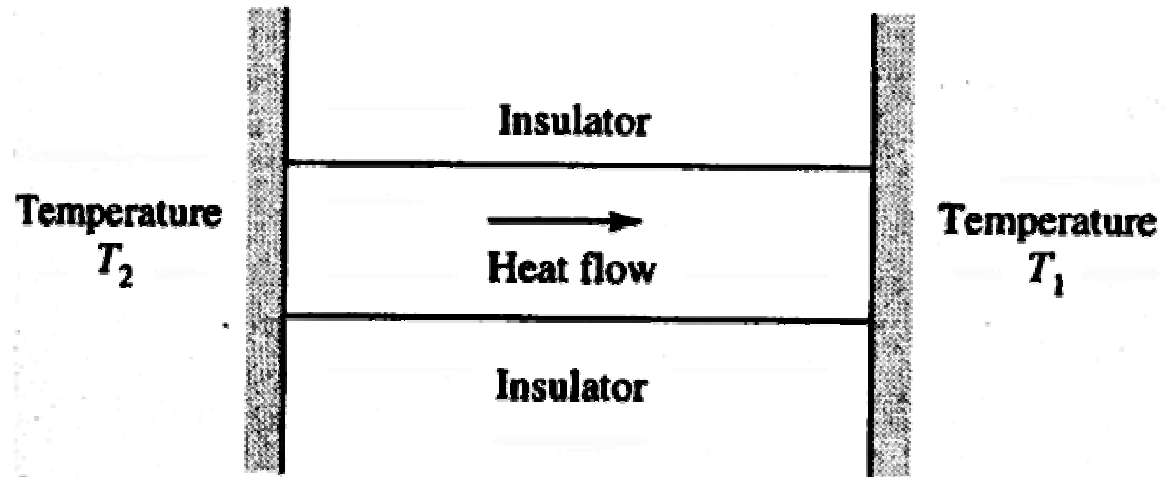where λ is the thermal conductivity of the bar. Express T_2 as the subject of the formula.

Figure 3.8 **The laterally insulated metal bar for Exercise 3.5**

Chapter 4 POWERS AND LOGARITHM

4.1 INTRODUCTION

Just as multiplication by a whole number can be thought of as repeated addition, for example:

$$5 \times 4 = 4 + 4 + 4 + 4 + 4$$

'raising to the power of' can be thought of as repeated multiplication. For instance, 4 raised to the power 5 gives

$$4^5 = 4 \times 4 \times 4 \times 4 \times 4$$

Here 4 is the base and 5 is the exponent, power or index (plural indices).

4^5 is read as '4 to the power 5'. If the index is a 2, e.g 4^2, this can be read as '4 to the power 2' or '4 squared'. If the index is a 3, e.g 4^3, this can be read as ' 4 to the power 3' or '4 cubed'.

In this chapter we look at extending the idea of powers so that the index does not necessarily need to be a whole positive number. We look at the properties of powers and the 'inverse' operations of roots and logarithms.

4.2 THE RULES OF INDICES

The rules of indices can be justified quite easily for simple cases. The main rules are listed below with an example. Here a can be any positive number.

Multiplication - add indices

$$a^m \times a^n = a^{m+n}$$

Example 4.1

$$2^3 \times 2^2 = 2^5$$

because

$$2^3 \times 2^2 = (2 \times 2 \times 2) \times (2 \times 2) = 2^5$$

Division - subtract indices

$$\frac{a^m}{a^n} = a^{m-n}$$

Example 4.2

$$\frac{3^5}{3^2} = 3^3$$

because

$$\frac{3^5}{3^2} = \frac{3 \times 3 \times 3 \times 3 \times 3}{3 \times 3}$$

$$= \frac{3 \times 3 \times 3 \times 3 \times 3}{1 \times 3 \times 3} = 3 \times 3 \times 3 = 3^3$$

Negative indices

$$a^{-n} = \frac{1}{a^n}$$

Example 4.3

$$2^{-3} = \frac{1}{2^3}$$

This takes a few steps to justify. Compare the following:

$$2^4 \times 2^{-3} = 2^{4+(-3)} = 2^1 = 2 \qquad\qquad (4.1)$$

(by using the multiplication rule) and

$$2^4 \times \frac{1}{2^3} = \frac{2^4}{2^3} = 2^{4-3} = 2^1 = 2 \qquad\qquad (4.2)$$

(by using the division rule). Hence, comparing Eq. (4.1) and Eq. (4.2)

$$2^4 \times 2^{-3} = 2^4 \times \frac{1}{2^3}$$

so

$$2^{-3} = \frac{1}{2^3}$$

Fractional indices

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

Example 4.4

$$2^{\frac{1}{3}} = \sqrt[3]{2}$$

Compare the following:

$$2^{\frac{1}{3}} \times 2^{\frac{1}{3}} \times 2^{\frac{1}{3}} = 2^{\left(\frac{1}{3}+\frac{1}{3}+\frac{1}{3}\right)} = 2^1 = 2$$

(using the multiplication rule) and

$$\sqrt[3]{2} \times \sqrt[3]{2} \times \sqrt[3]{2} = 2 \qquad (4.3)$$

by the definition of the cubed root. Hence

$$2^{\frac{1}{3}} \times 2^{\frac{1}{3}} \times 2^{\frac{1}{3}} = \sqrt[3]{2} \times \sqrt[3]{2} \times \sqrt[3]{2} = 2 \qquad (4.4)$$

Comparing Eq. (4.3) and Eq. (4.4) gives

$$2^{\frac{1}{3}} = \sqrt[3]{2}$$

The zeroth power

$$a^0 = 1$$

Example 4.5

$$2^0 = 1$$

Compare the following:

$$2^0 \times 2^2 = 2^{0+2} = 2^2$$

(by the multiplication law) and

$$1 \times 2^2 = 2^2$$

Comparing Eq. (4.5) and Eq. (4.6) gives $2^0 \times 2^2 = 1 \times 2^2$ so

$$2^0 = 1$$

Taking the power of a power - multiply indices

$$(a^m)^n = a^{mn}$$

Example 4.6

$$(4^2)^3 = 4^6$$

because

$$(4^2)^3 = (4{\times}4)^3 = (4{\times}4){\times}(4{\times}4){\times}(4{\times}4) = 4^6$$

A common error is to attempt to combine an addition of powers into a single power. This cannot usually be done. For example

$$2^3 + 2^2$$

cannot be directly expressed as a power of 2.

$$2^3 + 2^2 = 8 + 4 = 12$$

and 12 is not an exact power of 2.

However, if there is repetition of the same term then they can be combined by using the fact that multiplication is repeated addition:

$$2^2 + 2^2 + 2^2 + 2^2 = 4{\times}2^2 = 2^2 \times 2^2 = 2^4$$

Example 4.7 Express as a single power $(2^3)^2 \times 2^{-2}$

SOLUTION

$$(2^3)^2 \times 2^{-2} = 2^6 \times 2^{-2} = 2^4$$

Example 4.8 Express as a single power

$$\frac{\left(4^3\right)^{\frac{1}{2}}}{4^{\frac{1}{2}}} + 4$$

SOLUTION

$$\frac{\left(4^3\right)^{\frac{1}{2}}}{4^{\frac{1}{2}}} + 4 = \frac{4^{\frac{3}{2}}}{4^{\frac{1}{2}}} + 4 = 4^{\frac{3}{2}-\frac{2}{2}} + 4$$

$$= 4^1 + 4 = 2 \times 2^2 = 2^3$$

The rules we have looked at for indices apply to any positive base. The power of negative base is not always defined (amongst real numbers). For instance, try on your calculator to calculate $(-3)^{1/4}$ and you will find that it answers something like '-E-' which is the response it gives to an undefined operation. $(-3)^{1/4} = \sqrt[4]{(-3)}$ and this means find a number which when raised to the fourth power gives -3. We already have discussed that the square of any number is always positive, and this is also true for any other even power. It is not possible to find a number which raised to the fourth power would be negative, so $(-3)^{1/4}$ is not defined.

There are two powers that get an individual mention on a scientific calculator. That is 10^x and e^x. The reason for 10^x being of particular importance clearly comes from the fact that we perform day-to-day arithmetic to base 10. $e = 2.7182818$ to 7 decimal places, and its special relevance comes from its use in solving problems of exponential growth and decay which is discussed in Chapter 8 of Mathematics for Electrical Engineering and Computing.

4.3 THE INVERSE OPERATIONS

In Chapter 1 we discussed subtraction as the inverse operation to addition and division as (almost) the inverse of multiplication (only almost, because of the problem with 0).

Before thinking about the inverse of taking a power, notice that

$$3^2 \neq 2^3$$

because $3^2 = 3 \times 3 = 9$ and $2^3 = 2 \times 2 \times 2 = 8$. In general,

$$a^b \neq b^a$$

If we swap round the numbers, it does not give the same result. It is this unfortunate fact that leads to there being two inverses. To understand this, it will be easier to think about some simple equations involving powers, and bring back person A and person B.

Person A proposes the problem to B. $x^4 = 625$ by saying " I have thought of a number and raised it to the power of 4, the answer I get is 625; what is the number I first thought of?"

Luckily B, whose mental arithmetic is not all that good, happens to have a calculator. B also knows that the 'inverse' of raising to the power of 4 is taking the 4th root and calculates $\sqrt[4]{625}$ by using the fact that this is the same as $625^{1/4}$ This gives her the answer 5, but from experience with square roots (in Chapter 3) B knows to check for the possibility that -5 is also a solution. Sure enough, both 5^4 and $(-5)^4$ would result in 625.
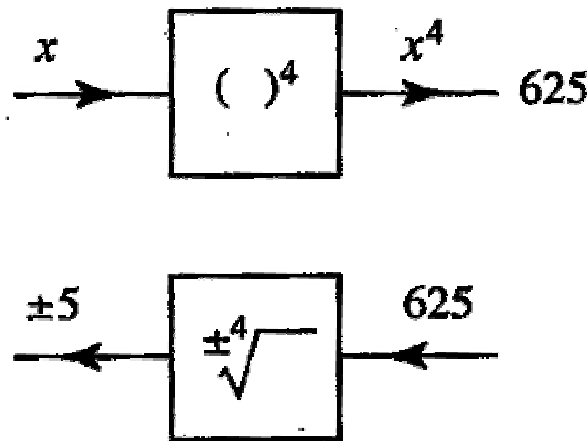
Figure 4.1. The equation $x^4 = 625$ pictured using a flow diagram. The inverse flow diagram is found by moving in the opposite direction replacing the operation by its 'inverse' and this leads to the solution $x = \pm\ 5$.
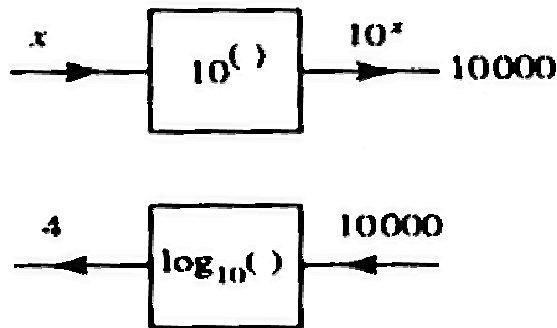
This problem can be pictured using flow diagrams as used in Chapter 3, and is shown in Figure 4.1.

B then announces to A that the number she first thought of was 5 or -5. A accepts this answer and goes on to propose a new problem. $4^x = 1024$, by saying " I have thought of a number and raised 4 to the power of this unknown number. The answer I get is 1024. What is the number I first thought of?"

Notice that the unknown is now the exponent, not the base. The inverse in this case is different. B, of course checks this by trying out the 4th root. The 4th root of 1024 is 5.6569 to 4 decimal places, but this certainly does not satisfy the equation, as $4^{5.6569} = 2545$. B needs a new operation to solve the problem. The operation needed is called the logarithm, base 4. The answer to the problem is $\log_4 (1024)$. Unfortunately, B is still stuck because this operation is not on her calculator. She only has a choice of log, which is shorthand for the logarithm base 10, or ln, which is shorthand for the logarithm base e.

B explains her dilemma to A, who therefore proposes the simpler problem, $10^x = 10000$ by saying 'I have thought of a number and raised 10 to the power of this unknown number. The answer I get is 10000, what is the number I first thought of?'

B is now able to answer this problem by finding $\log_{10}(10000)$, giving the answer 4. This problem and its solution are pictured in a flow diagram in Figure 4.2. B checks that $10^4 = 10\ 000$, which is correct, and is able to give A the answer. However B still has some work to do to find the answer to the original problem of $4^x = 1024$. We will return to this later.

Figure 4.2 The equation $10^x = 10\ 000$ pictured using a flow diagram. The inverse flow diagram is found by moving in the opposite direction replacing the operation by its inverse. This leads to the solution $x = 4$.

These problems, which result in us needing logarithms, lead us to a definition of the logarithm. If some number y is the result of raising a to the power x, i.e.

$$y = a^x \tag{4.7}$$

then the relationship between x and y can also be expressed as

$$x = \log_a(y) \tag{4.8}$$

That is, $\log_a$ is defined as an inverse operation.

Another idea about inverses was that if one operation is performed on some number, then the inverse operation, we should get back to the number we first started with. For instance, in Chapter 1 we started with 5, added 3, then subtracted 3 (the inverse of adding), which took us back to 5. Similarly if we start with 4, find 10^4, and then take $\log_{10}$, we should get back to 4. Check this on the calculator. This means that

$$\log_{10}(10^4) = 4$$

and in general, replacing 4 by any number, x, and using a to represent any base, we get

$$\log_a(a^x) = x.$$

Here a should be a positive number else things can get tricky. x can take any value.

This will also work the other way round: if you take the $\log_a$ first and raise a to the power of the result you should get back to your original number. Hence

$$a^{\log_a(x)} = x$$

However, this way round, not only should a be positive but also x must be a positive number. The logarithm of 0, and logarithms of negative numbers are not defined. Your calculator responds with the error condition '-E-' if you try to calculate the logarithm of a negative number.

As logarithms and powers are so closely linked, not surprisingly there are rules for logarithms that match each of the rules given in Section 4.2

4.4 RULES OF LOGARITHMS

Check the example in each case.

Log of a product - add logs

$$\log_a(xy) = \log_a(x) + \log_a(y)$$

Example 4.9

$$\log_{10}(10 \times 100) = \log_{10}(10) + \log_{10}(100)$$

Log of a division - subtract logs

$$\log_a\left(\frac{m}{n}\right) = \log_a(m) - \log_a(n)$$

Example 4.10

$$\log_{10}\left(\frac{1000}{100}\right) = \log_{10}(1000) - \log_{10}(100)$$

Log of a reciprocal

$$\log_a\left(\frac{1}{x}\right) = -\log_a(x)$$

Example 4.11

$$\log_{10}\left(\tfrac{1}{10000}\right) = -\log_{10}(10000)$$

The inverse rule

$$\log_a(a^x) = x$$

and it follows that

$$\log_a(a) = 1$$

Example 4.12

$$\log_{10}(10^4) = 4$$

The log of 1

$$\log_a(1) = 0$$

Example 4.13

$$\log_{10}(1) = 0$$

The log of a power

$$\log_a(x^n) = n\log_a(x)$$

Example 4.14

$$\log_{10}(100^4) = 4\log_{10}(100)$$

Example 4.15 Simplify

$$\log_{10}\left(\frac{100^2}{10^3}\right)$$

SOLUTION

$$\log_{10}\left(\frac{100^2}{10^3}\right) = \log_{10}(100^2) - \log_{10}(10^3)$$

$$= \log_{10}((10^2)^2) - 3\log_{10}(10)$$

$$= \log_{10}(10^4) - 3 = 4 - 3 = 1$$

The change of base rule

$$\log_a(x) = \frac{\log_b(x)}{\log_b(a)}$$

Example 4.16

$$\log_{100}(10000) = \frac{\log_{10}(10000)}{\log_{10}(100)}$$

Example 4.17 We are now able to solve the problem that A posed to B earlier: find x if $4^x = 1024$.

SOLUTION

Take the logarithm, base 4, of both sides giving

$$\log_4(4^x) = \log_4(1024)$$

as $\log_4(\ )$ is the inverse operation to $4^{()}$, the left hand side gives x, so we have

$$x = \log_4(1024)$$

Supposing our mental arithmetic is not up to calculating the value of the right hand side then we can use the change of base rule to find

$$\log_4(1024) = \frac{\log_{10}(1024)}{\log_{10}(4)}$$

And using a calculator this gives the answer 5.

4.5 SUMMARY

1. a^b is read as 'a raised to the power b'. It can be thought of as repeated multiplication when b is a whole positive number.
2. There are many rules for indices that are summarised in Section 4.2.
3. The 'inverse' of raising to the power of n is to take the nth root. If $y = x^n$ then $x = \sqrt[n]{y}$, providing x is positive.
4. The inverse of exponentiation is taking the logarithm. If $y=a^x$ then $x = \log_a(y)$, where a is positive.
5. There are many rules for logarithms, which are summarised in Section 4.4.

4.6 EXERCISES

4.1 Explain, using the idea of the inverse, why $\sqrt[n]{x^n} = x$, where x is a positive number and n $\neq 0$. Test this out on a calculator using various values of x and n. Investigate what happens if x is allowed to take negative values.

4.2   (a) Find $\left(-27a^{-2}\right)^{\frac{1}{3}}$ when a = 8.

       (b) Find $\left(3b^{-3}\right)^{-1}$ when b = 2

       (c) Evaluate $2^{-3}ab^2c^{-1}$ when a = 2, b = 3, c = 4.

       (d) Find $\left(a^3b^{-1}\right)^{\frac{1}{2}}$ when a = 3 and b = 2.

4.3 Evaluate:

       (a) $16^{\frac{3}{2}}3^{-1}$

       (b) $9^{\frac{3}{4}} \times 3^{-\frac{1}{2}}$ (c) $\dfrac{2^{\frac{1}{3}}2^{\frac{4}{3}}}{4}$

4.4 Express the following as single powers, where possible:

       (a) $\sqrt{a} \times a^3$ (b) $\dfrac{3^7 \times 3^2}{3^4}$ (c)$(2^4)^5$

       (d) $\dfrac{2^{18}}{2^{16}}$ (e)$\sqrt[3]{a^6b^9}$ (f) $(9a^2)^2$

       (g) $\sqrt{2^{16}}$ (h)10 000   (i) $\sqrt{y} \times \sqrt[3]{y}$

       (j) $y^2 + y^3$

4.5 Express as a single logarithm:

       (a) $\log(x^3) + 3\log(x)$ (b) $\log(2xy) - \log(x) - \log(y)$

       (c) $2\log(x^3) - \log(x^2) - 3\log(x)$

4.6 Use the change of base rule to express the following in terms of $\log_{10}$, and hence calculate their values:

      (a) $\log_8(6)$ (b) $\log_3(12)$ (c) $\log_4(1000)$

4.7 Solve the following equations:

      (a) $x^4 = 16$ b) $x^3 = -27$ (c) $x^6 = -64$

      (d) $x^{1/2} = 4$ (e) $x^{\frac{3}{5}} = 125$ (f) $\log_{10}(x) = 0$

      (g) $\log_2(x) = 3$ (h) $10^x - 1 = 0$ (i) $2\log_{10}(x) - 4 = 0$

4.8 A voltage-sensitive resistor has a current/voltage relationship of the form $I = kV^n$ where I is expressed in mA, V is in volts and k and n are constants. Find k and n if $I = $ 10mA when $V = 1.7$V and $I = 2$mA when $V = 1$V.

CHAPTER 5 MEASUREMENT AND CALCULATION

5.1 INTRODUCTION

Engineering measurements cannot be guaranteed to be 100 per cent accurate. The amount of possible inaccuracy is referred to as the maximum error in the measurement. Performing calculations can also introduce (hopefully small) errors into a result. All but the simplest of calculations are performed either by using a calculator or a computer and these can accommodate only a limited number of digits in each number. After each stage of the calculation the result may need to be shortened so that it can again fit into the number of digits available. For instance $1/3$ gives the result $0.3333333333....$ $(0.\dot{3}$ which reads as '0.3 recurring'), but a calculator typically gives the result 0.3333333, shortening the result to 7 decimal places. This shortening process introduces a rounding error.

The purpose of this chapter is to present ideas to be used in expressing errors in measurement and procedures that should be used to ensure a sensible numerical calculation. It is not our aim to discuss the details of which button to press on a calculator or in which order (for this you should refer to the operation manual provided). Many of the ideas concerning calculation are equally relevant to using a computer.

5.2 ROUNDING A SINGLE NUMBER

There are two common ways of rounding a single number: by expressing it to a specified number of decimal places or to a specified number of significant figures.

Decimal Places

Example 5.1   Write 2.45782 to two decimal places.

SOLUTION Count two places after the decimal point: this would give 2.45. The result of 2.45 is what we get if we chop the number after two decimal places. A more accurate way to express it is to round the number by looking at the next digit, in this case a 7. If the digit is equal to or greater than 5 then we round upwards, i.e. increase the previous digit by 1, while if it is less than 5 then we round downwards, giving the same answer as if we chopped the number after the second decimal place. In this case, we round upwards because 2.45782 is nearer to 2.46 than it is to 2.45 as can be seen in Figure 5.1. Thus 2.45782 = 2.46 to two decimal places.

Figure 5.1 2.45782 rounded to 2d.p. gives 2.46, as the number is nearer to 2..46 than to 2.47

Example 5.2 Write 49.203 to two decimal places.

SOLUTION Count two places after the decimal point: this would give 49.20 = 49.2.
Looking at the next digit, in this case a 3 we see that we do not need to round upwards.
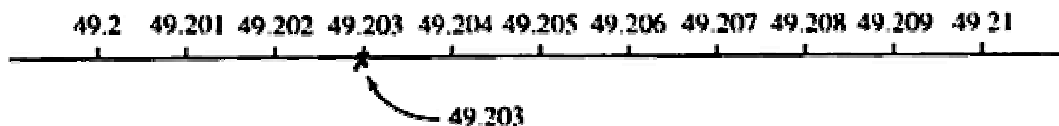Thus 49.203 = 49.2 to two decimal places (see Fig. 5.2).

Figure 5.2 49.203 rounded to two decimal places gives 49.2 as the number is nearer to 49.2 than to 49.21.

Significant figures

Example 5.3 Write 2.45782 to four significant figures.

SOLUTION Count four figures from the left of the number, ignoring any leading zeros;
this would give 2.457. The result of 2.457 is what we get if we chop the number after four
significant figures. Again we round the number by looking at the next digit, in this case
an 8. If the digit is equal to or greater than 5 then we round upwards, but if it is less than
5 then we round downwards. In this case we round upwards because 2.45782 is nearer to
2.458 than it is to 2.457. Thus 2.45782 = 2.458 to four significant figures.

Example 5.4 Write 0.00049203 to two significant figures.

SOLUTION Count two figures from the left of the number, ignoring any leading zeros:
this would give 0.00049. Again we round the number by looking at the next digit, in this
case a 2. As this is less than 5 we round downwards to give 0.00049. Thus 0.00049203 =
0.00049 to two significant figures.

5.3 ERROR FROM ROUNDING

By rounding numbers as in section 5.2 we introduce an error into the result. This
error can be studied by looking at the greatest and least possible values of a quantity
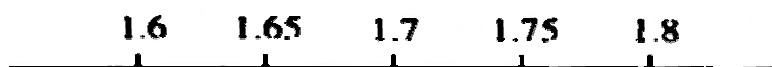expressed to a specified number of significant figures.

Example 5.5 A value is given as 1.7 to two significant figures. What are the
greatest, and least values that it could take? What is the maximum error in the given
value?

SOLUTION Consider a number line near the value of 1.7, as in Fig. 5.3, marking numbers with up to two significant figures and halfway points between them. To be nearer to 1.7 than 1.6 the number must be just greater than 1.65. To be nearer to 1.7 than 1.8 then it must be less than 1.75. The greatest value it can take is (just less than) 1.75 and the least value is 1.65. The maximum error can be found by halving the range of the possible values, that is:

$$\text{range of possible values: } 1.75 - 1.65 = 0.1$$

$$\text{maximum error} = \text{range}/2 = 0.1/2 = 0.05$$

The value could be given as 1.7 ± 0.05

Figure 5.3 A value is given as 1.7 to two significant figures. It must therefore lie in the range 1.65 to 1.75

Example 5.6 The velocity of light is given as 300 000 000 ms^{-2} to three significant figures. What are the greatest and least values it could be?

SOLUTION Consider a number line around the value of 300 000 000, as in Fig. 5.4, marking numbers with up to 3 significant figures and halfway points between them. The least and greatest values it could be are 299 500 000 and (just less than) 300 500 000.

$$\text{Range of possible values} = 300\ 500\ 000 - 299\ 500\ 000 = 1\ 000\ 000$$

$$\text{Maximum error} = \text{range}/2 = 1\ 000\ 000/2 = 500\ 000$$

The value could be given as $300\ 000\ 000 \pm 500\ 000$

Figure 5.4 **The velocity of light is given as 300 000 000 to three significant figures. It must therefore lie in the range 299 500 000 to 300 500 000**

5.4 ERROR FROM MEASUREMENT

Consider using a ruler to measure the size of a table. The smallest measure on the ruler is a millimetre; hence the result is only likely to be correct to the nearest millimetre and probably (as you may have moved slightly while taking the measurement) even less accurate than that. If we used the same ruler to measure the length of a football pitch then

errors would build up at each stage and we wouldn't expect the answer to be correct to the last millimetre and not even to the last centimetre.

The error in a calculation or measurement cannot be known exactly. In the absence of any clear idea of how to estimate the error then one way to proceed is to repeat the measurement several times. Then take the average of all the measurements as the measured value and the range of values gives an idea of the possible maximum error. More detailed methods of analysing the error are looked at in Chapter 21 of Mathematics for Electrical Engineering and Computing on Probability and Statistics.

Suppose that measuring the table leads to the values 1.48m, 1.46m, 1.47m, 1.49m, 1.495m. The average of the values gives 1.479m. The range of values is 1.495 - 1.46 = 0.035, giving an estimate of the error as 0.035/2 = 0.0175m. Therefore the length of the table is 1.479 ± 0.0175m. The error expressed in this form is called the *absolute* error. If it expressed as a fraction or percentage of the true value then it is called the *relative* error. Using the average value as an estimate of the true value gives the percentage error as

(0.0175/1.479) x 100 = 1.2%

Then the length of the table is given as 1.479 ± 1.2% and the relative error is ± 1.2%.

Sometimes the possible errors can be found by using known information about the measuring apparatus.

If we measure the current flowing in a circuit using an ammeter then the presence of the ammeter increases the resistance in the circuit. This in turn will introduce an error into the reading. This error cannot be avoided, but it can be kept to a minimum by choosing an ammeter whose resistance is very much less than that of the circuit, say less than 1% of the circuit resistance. The error introduced by the ammeter is proportional to the size of the current being measured and also to the ratio of the internal resistance of the ammeter to the circuit resistance (Fig. 5.5).
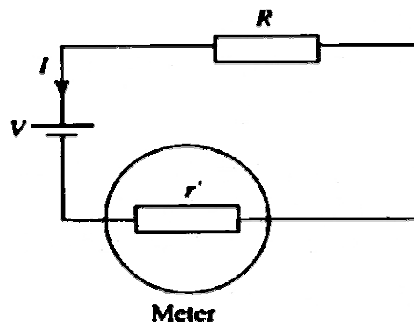

Figure 5.5 Using a meter to measure the current in a circuit. *R* is the resistance of the circuit and r' is the internal resistance of the ammeter.

If the meter is chosen such that the internal resistance is very much less than the resistance to be measured then the effect of this error cannot be too serious. For instance, if the circuit resistance is known to be around 10 KΩ then a meter with an internal resistance of about 100 Ω will introduce an error of about $100/10000 = 1/100 = 1\%$.

It is usually true that small errors are roughly proportional to the size of the quantity being measured. For this reason it is more useful to indicate the relative error. In the example just considered, suppose we measure the current to be 12.4mA. As we think the meter has introduced an error of up to 1% (in excess), and considering other possible errors in the measurement (caused, for instance from the presence of magnetic fields, friction of moving parts, error in reading from the meter) as introducing smaller errors then we could adjust the reading down by 1% to take account of the effect of the internal resistance and estimate the error as being of the order of 1%. This gives an estimate of the current as 12.4 mA - 1% of 12.4 mA = 12.4 - 0.124 = 12.276 mA. This gives the current as $12.276 \pm 1\%$ (or 12.276 mA ± 0.123 mA or between 12.153 mA and 12.399 mA)

If this measurement is now used to calculate other circuit values then the final result should only be expressed to a consistant level of accuracy. As the original measurement is correct to no more than three significant figures then the final calculated result should be expressed to no more than three significant figures, with an indication that the maximum error will be at least as big as the original error.

Example 5.7 An object is weighed on 4 different scales giving the values 1.01kg, 0.99kg, 0.97kg, 1.02kg. Estimate the weight of the object, indicating an error range.

SOLUTION Take the average of the given values for the weight. This gives weight $= (1.01 + 0.99 + 0.97 + 1.02)/4 = 0.9975$

The range of values is 1.02 - 0.97 = 0.5, giving an estimate of the maximum error as 0.5/2 = 0.25. This gives the weight of the object as 0.9975 ± 0.25 kg.

Example 5.8 A voltmeter with an internal resistance of 1 MΩ is used to measure the voltage across a 5 KΩ resistor (See Fig. 5.6). The voltage recorded is 20.1 V, estimate the voltage across the resistor.
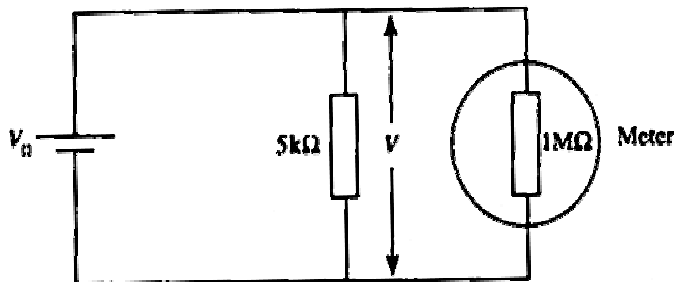
Figure 5.6 Using a voltmeter to measure the voltage across a 5 kΩ resistor where the internal resistance of the voltmeter is 1MΩ.

SOLUTION A voltmeter will underestimate the voltage by an amount given by the ratio of the resistance of the resistor divided by the internal resistance of the voltmeter. Therefore a better estimate of the resistance is the measured value 20.1 - (5000/1 000 000) of 20.1. This gives 20.1 - 0.5% = 20.1 - 0.1005 = 19.9995. Therefore we could give the voltage as 19.9995 $\pm$ 0.5% = 19.9995 $\pm$ 0.1005 = 19.899 to 20.1 V. This also takes account of other possible errors in the measurement, as referred to when considering the ammeter.

5.5 SCIENTIFIC NOTATION

The display of most calculators is limited to about eight digits. This therefore presents a problem when trying to enter a number of smaller magnitude than 0.000 000 1 or bigger than 999 999 99. To overcome this problem scientific notation is used. The number 105000000 can be represented as 1.05 x 10^8 and in this form the number can be entered into the calculator.

A number is in scientific notation when it is expressed as a number between 1 and 10 (ie with only one digit before the decimal point) multiplied by some power of 10. To calculate the correct power of 10 count the number of places that the decimal point has moved.

Example 5.9 Express 10 873.5 in scientific notation

SOLUTION

$$10873.5 = 1.08735 \text{ x } 10^4$$

To move the decimal point from its original position to after the first digit requires a movement of four places. The power must be positive because 1.08735 is smaller than 10873.5

Example 5.10 Express 0.0000563289 in scientific notation

SOLUTION

$$0.0000563289 = 5.63289 \text{ x } 10^{-5}$$

To move the decimal point from its original position to after the first non-zero digit requires a movement of five places. The power must be negative because 5.63289 is bigger than 0.0000563289.

Machines store floating point numbers in a manner similar to scientific notation. Machine numbers are expressed as binary (base 2) numbers rather than decimal numbers

(base 10), which we use for hand calculations. However they still have a limited amount of space to store the number. The result of a calculation on most calculators can only have an accuracy of seven or eight significant figures at the most. Most computer calculations have a similar accuracy.

Another reason for using scientific notation is to aid checking a calculation. This we look at in the section 5.7.

5.6 DIMENSIONS AND UNITS

The dimensions of a physical quantity give the way it is related to the fundamental quantities of mass, length, time, electric current, temperature, amount of substance and luminous intensity. The metric units for these are given in the Systeme International d'Unites (SI) as:

Quantity	Unit	Symbol
Length	metre	m
Mass	kilogram	kg
Time	second	s
Electric current	ampere	A
Temperature	kelvin	K
Amount of substance	mole	mol
Luminous intensity	candela	cd

From these other units can be derived, some of the most common being:

Quantity	Unit	Symbol
Force	newton	$N = kg\ ms^{-2}$
Work, energy, heat	joule	$J = N\ m$
Power	watt	$W = J\ s^{-1}$
Electric charge	coulomb	$C = A\ s$
Electric potential	volt	$V = W\ A^{-1}$
Electric resistance	ohm	$\Omega = V\ A^{-1}$
Inductance	henry	$H = V\ s\ A^{-1}$
Frequency	hertz	$Hz = s^{-1}$
Magnetic Flux	weber	$Wb = V\ s$
Magnetic flux density	tesla	$T = Wb\ m^{-2}$

There are other ways of expressing these 'compound units'; for instance, one newton = $kg\ m\ s^{-2}$ can be written as $kg\ m/s^2$

And, is read as 'kilogramme metres per second per second' or 'kilogramme metres seconds to the minus 2'.

When performing a calculation the units of the result can be found from the units of the values in the formula. For instance if velocity = distance/time, then given that the distance travelled is 20m in 5s then velocity = 20m/5 s = 4ms^{-1} or 4m/s (4 metres per second).

Notice that we performed the calculation on the units and this must give the correct units for the quantity being calculated. If a formula contains an addition or a subtraction then each part of the formula must have the same units. For example, if an object is moving under constant acceleration then the distance travelled, s, can be related to the time t, initial velocity u, and the acceleration a, by the formula:

$$s = ut + \tfrac{1}{2}\,at^2$$

if $u = 12$ m s^{-1}, $t = 5$s , $a = 6$ m s^{-2} then

$$s = 12 \times 5(\text{m s}^{-1}\times\text{s}) + 6 \times 5 \times 5(\text{m s}^{-2}) \times \text{s} \times \text{s}$$

$$= 60 \text{ m} + 75 \text{ m} = 135 \text{ m}$$

Notice that both parts of the formula, ut, and $\tfrac{1}{2}\,at^2$ give units of metres, which is correct, as it is a distance that we are calculating.

Prefixes indicating multiples and sub-multiples of units

The most common of these are:

Multiplication Factor	Prefix	Symbol
10^9	giga	G
10^6	mega	M
10^3	kilo	k
10^{-2}	centi	c
10^{-3}	milli	m
10^{-6}	micro	μ
10^{-9}	nano	n
10^{-12}	pico	p

5.7 ROUGH CALCULATIONS

Rough calculations should be used as a way of checking longer calculations either by hand or on a calculator. First, write all the numbers in scientific notation. Separate out all the powers of 10. Approximate the numbers, possibly to only one significant figure or some other convenient amount and perform the calculation by hand or mentally. If the rough calculation is more or less the same as the value found from using the calculator then it can be assumed that the calculated value is correct.

Example 5.11 The conductivity of a length of conducting material of uniform cross section can be obtained from the expression

$$\sigma = \frac{IL}{VA}$$

where, I is the current, L is the length of the section, A is the cross-sectional area and V is the voltage drop across the material.

A certain section of conducting material in an integrated circuit device has a current passed across it of 0.004 A, producing a voltage drop of 84 mV. The length is 3.5mm and it has a cross-section that is a rectangle of dimensions 1 micrometre by 3.7 micrometres. Determine its conductivity.

SOLUTION. First, write all the quantities in consistent units and in scientific notation.

$$I = 4 \times 10^{-3} \text{ A}$$

$$L = 3.5 \times 10^{-3} \text{ m}$$

$$V = 84 \times 10^{-3} \text{ V} = 8.4 \times 10^{-2} \text{ V}$$

$$A = 1 \text{ x } 10^{-6} \text{ x } 3.7 \text{ x } 10^{-6} \text{ m}^2$$

This gives

$$\sigma = \frac{4 \times 10^{-3} \times 3.5 \times 10^{-3}}{1 \times 10^{-6} \times 3.7 \times 10^{-6} \times 8.4 \times 10^{-2}} \frac{\text{Am}}{\text{Vm}^2}$$

This can be approximated to

$$\frac{4 \times 4}{1 \times 4 \times 8} \times \frac{10^{-3-3}}{10^{-6-6-2}} = = \frac{1}{2} \times \frac{10^{-6}}{10^{-14}} = 0.5 \times 10^8 = 5 \times 10^7$$

The result, using a calculator, gives $4.504\ 504 \times 10^7$. The approximate result confirms that the result given using a calculator is a sensible one. The units are:

$$\frac{\text{A m}}{\text{V m}^2} = \frac{\text{A}}{\text{V m}} = \frac{1}{\Omega \text{m}} = (\Omega \text{ m})^{-1}$$

However, we should finally notice that the calculator has produced a result with seven significant figures in the result. This is unreasonable considering we should assume that the original numbers had no greater accuracy than two significant figures. We therefore represent the result to two significant figures, giving the conductivity as 4.5 x 10^7 $(\Omega \text{ m})^{-1}$

5.8 PERFORMING CALCULATIONS ON A CALCULATOR

Each stage in a calculation may introduce a rounding error into the result. The larger the number of significant figures used to perform the calculation the smaller the rounding error will be. The best accuracy is obtained by performing all calculations to the maximum number of significant figures available on the calculator. Even in the case where our original values were only accurate to, say, three significant figures, we should perform calculations to the maximum number of significant figures and then round at the end to a number of figures consistent with our original measurements. To see an example of the problem introduced by large rounding errors, perform a calculation like (9.1111/9) x 9

We can see immediately that as we are dividing a number by 9 and then multiplying again by 9 we should get back to the original number of 9.1111. Now perform this on a calculator, writing down any interim results to the maximum number of significant figures.

9.1111/9 = 1.0123444, to eight significant figures.

Multiply this by 9 to get 9.1110996, which, rounded to five significant figures gives 9.1111, the correct result. Now do the same thing again but only writing the interim result to five significant figures

$$9.1111/9 = 1.0123, \text{ to five significant figures.}$$

Multiply this by 9 to get 9.1107, which is no longer correct to five significant figures.

A relatively large error has been introduced by not using the largest possible number of significant figures in the course of doing the calculation.

Use of the memory on the calculator can prevent the need to write down a lot of long numbers. However, it is a good idea to write down the result after each four or five operations in order to be able to go back and check the result, should it appear not to be correct.

Example 5.12 Three capacitors are in series in a circuit (Fig.5.7). The Equivalent capacitance is given by

$$C = \frac{C_1 C_2 C_3}{C_1 C_2 + C_2 C_3 + C_1 C_3}$$
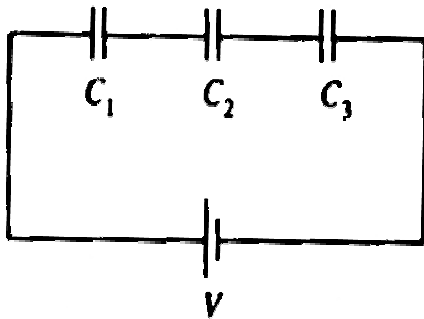
Figure 5.7 Three capacitors in series

If the three capacitors have capacitances of $C_1 = 4.2\,\mu\text{F}$, $C_2 = 5.6\,\mu\text{F}$ and $C_3 = 2.7\,\mu\text{F}$ then calculate the equivalent capacitance C.

SOLUTION

$$C = \frac{4.2 \times 5.6 \times 2.7}{4.2 \times 5.6 + 5.6 \times 2.7 + 4.2 \times 2.7} x \frac{\mu\text{F} \times \mu\text{F} \times \mu\text{F}}{\mu\text{F} \times \mu\text{F}}$$

An approximate calculation gives

$$\frac{4 \times 6 \times 3}{4 \times 6 + 6 \times 3 + 4 \times 3} \mu F = \frac{72}{54} \mu F \approx 1.3 \mu F$$

This approximate result confirms that the result given by the calculator is a sensible one. However, a result with eight significant figures does not seem reasonable in the circumstances, as we could assume the original measurements are only accurate to two significant figures. Finally then, we round the result to two significant figures giving $1.3\mu F$ for the equivalent capacitance.

Example 5.13 The quantity of heat that will raise a substance from temperature T_1 °C to T_2 °C is given by

$$Q = mc(T_2 - T_1)$$

joules, where m is the mass of the substance (given in kg), c is the specific heat capacity in J kg^{-1} °C^{-1}. A mass of 3.2 kg of aluminium absorbs 30 405 joules of heat and has a final temperature of 30°C. Find the original temperature of the block. (The specific heat capacity of of aluminium is 950J kg^{-1} °C^{-1} for temperatures around room temperature.)

SOLUTION Rearranging the expression for the heat absorption so that the original temperature is the subject of the formula gives

$$T_1 = T_2 - \frac{Q}{mc}$$

$$= 30 - \frac{30450}{3.2 \times 950} \text{ °C} = 30 - 10.016447 \text{ °C} = 19.983553 \text{ °C}$$

A rough calculation gives

$$T_1 = 30 - \frac{30000}{3x1000} \text{ °C} = 30 - 10 = 20 \text{ °C}$$

The rough calculation gives a figure near the calculated value therefore indicating that the value is reasonable. Finally, we look at the accuracy of the original values. They are apparently correct to two significant figures, so we round the value of the initial temperature of 19.983553 to two significant figures giving 20 °C.

5.9 SUMMARY

1. A number can be approximated, by expressing it to a specified number of significant figures or decimal places.
2. Errors are introduced into a result from errors in measurement and rounding errors.

3. Any measurement should be accompanied by an idea of the possible error in the measurement.
4. Scientific notation allows the representation of large and small numbers using only a limited number of digits.
5. Rough calculations should be used to check any calculated results.
6. To keep the effect of rounding errors to a minimum, always record interim calculations to the maximum number of significant figures or use the memory on the calculator.
7. The final value given after a calculation should be rounded to a number of significant figures consistant with the initial measurement(s).

5.10 EXERCISES

5.1 Express the following to the number of decimal places (d.p.) indicated:

(a) 2.567 (2 d.p.)   (b) 105 879.225 (1 d.p.)

(c) 0.000007 (1 d.p.)   (d) 38.99999 (2 d.p.)

5.2 Express the following to the number of significant figures (s.f.) indicated:

(a) 23.846 (3 s.f.)   (b) 10 456 879.22 (5 s.f.)

(c) 1.00056 (2 s.f.)   (d) 0.000000056789 (2 s.f.)

5.3 The following numbers are approximate. Estimate the maximum and least possible values they could be and express the maximum error as a percentage.

(a) 1.6   (b) 0.355

(c) 0.000046   (d) 2 100 000 000

5.4 The distance between two cities is recorded as 270 kilometres. What are the greatest and least possible values for the distance?

5.5 The following measurements are given with an indicated percentage error. Give the greatest and least possible values of the measured value.

(a) 5.1 $\Omega \pm 3\%$ (b) 0.0061 s $\pm 0.5\%$

(c) 12.9 kg $\pm 5\%$ (d) 583 000 m $\pm 4\%$

5.6 The following are sets of recorded measurements of the same quantity. Give an estimate of the value and an error range, also expressing the maximum error as a percentage.

(a) The length of a metal bar 12.2 cm, 11.95 cm, 12.05 cm, 12.15cm, 11.9 cm.

(b) A force: 5.12 N, 5.3 N, 5.6 N, 5 N.

(c) The settling time of an electric circuit after the power supply has been switched on 0.12s, 0.1 s, 0.15 s , 0.143 s , 0.11 s.

5.7 Write the following numbers in scientific notation:

(a) 12 000 000   (b) 0.0000007869   (c) 46.789   (d) 9.0005 8.

5.8 Calculate the following, checking each calculation by an approximation and writing the answer to an appropriate number of significant figures:

(a) 49.2 + 5.21 + 8.64 - 12.2

(b) $\dfrac{8.7x9.6}{12} - \dfrac{6.2-1.4}{9.6}$

(c) $\dfrac{(16.27-0.782)13.21}{19.46+17.21}+12.68$

(d) $\dfrac{0.0078}{360000} + \dfrac{0.0063}{720000} + \dfrac{6.8}{25}$

5.9 A resistor, R_1, is quoted as 10 Ω with 10% tolerance and another resistor, R_2, is quoted as 25 Ω with 5% tolerance. When in parallel the combined resistance is given by R, where

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

Calculate R and the tolerance for R when R_1 and R_2 are arranged in parallel.

5.10 The resistance of a length of copper wire is given by

$$R = \frac{pl}{a}$$

Where p is the resistively of copper = 1.7 x 10^{-8} Ωm at room temperature, l = length of wire and a = Cross sectional area of the wire= πr^2 ; r is the radius of the wire.

Find the resistance of a piece of copper wire of length 0.2m and diameter 1mm at room temperature.

5.11 The manner in which the resistance of any material varies with temperature is approximately given by the formula $R = R_0 (1 + \alpha T)$, where T is the temperature in $^{\circ}$C, α is the temperature coefficient of resistance and R_0 is the resistance at $0\ ^{\circ}$C. α for carbon is -0.00048 $^{\circ}$C^{-1}. If a length of carbon wire has a resistance of 0.01 Ω at 20 $^{\circ}$C, find its resistance at 0 $^{\circ}$C.

CHAPTER 6 RIGHT ANGLED TRIANGLES

6.1 INTRODUCTION

Right-angled triangles are triangles in which one angle is 90°, that is one quarter of a complete rotation. If the length of one side and the size of one angle (as well as the right angle) are known, or if two of the sides are known, then any other angle or side can be calculated. The ratios of any two sides of a right- angled triangle are called trigonometric ratios. The most important of these are the sine, cosine and tangent. The value of the ratios is found to depend only on the angles in the triangle; hence they can be tabulated against the angles and found using a calculator. Pythagoras's theorem relates the lengths of the sides with each other and is used to find a third side if two of the lengths are known.

Right-angled triangles are important for understanding vectors and complex numbers (see Chapters 9 and 10 of Mathematics for Electrical Engineering and Computing). Often other shapes can be split into a number of right-angled triangles and the properties of these used to find lengths and angles in the more complex shapes.

6.2 The angle sum of a triangle

The sum of the internal angles of a triangle is 180°. This means that if we have a right-angled triangle and we know the value of one of the angles then the third angle can be found. (See Fig. 6.1). In 6.1(a), $20° + 90° + x = 180° \Leftrightarrow x = 180° - 90° - 20° \Leftrightarrow x = 70°$. In (b), $40° + 90° + x = 180° \Leftrightarrow x = 180° - 90° - 40° \Leftrightarrow x = 50°$. In (c), $65° + 90° + x = 180° \Leftrightarrow x = 180° - 90° - 65° \Leftrightarrow x = 35°$
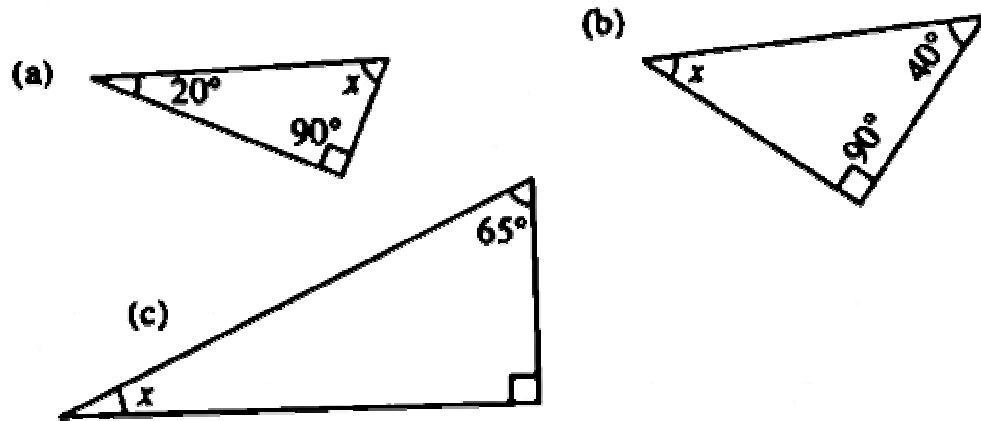
Figure 6.1 The angle sum of a triangle is 180⁰

Note for right-angled triangles the same result can be found by subtracting the other known angle from 90°. Thus: **(a)** $x = 90° - 20° = 70°$ **(b)** $x = 90° - 40° = 50°$ **(c)** $x = 90° - 65° = 35°$

6.3 SIMILAR SHAPES

Objects are similar if they are the same shape but not necessarily the same size (Fig. 6.2). Similar shapes will also have the same angles.

A similar shape is found by performing a change of scale: by magnifying or miniaturising. If, for instance, one side is doubled in length then so are all the others. The new shape so formed will be similar to the original. As the sides have been magnified or miniaturised by the same amount then there can be no change in the relative lengths of any two sides. The ratios of the sides are not changed by magnification or miniaturisation.

Similar triangles can be nested to show that the corresponding angles are equal (see Fig. 6.3). As any two right-angled triangles containing the same angle must be similar then the ratios of any two sides in a right-angled triangle can be found if we know the angles in the triangle.
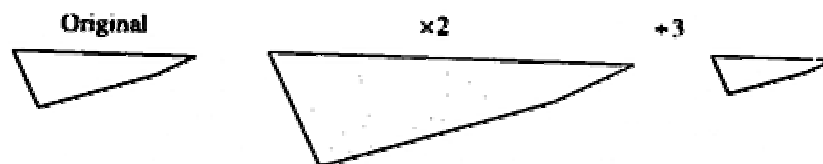
Figure 6.2 These shapes are similar. Notice that corresponding angles are equal. Also notice that the ratio between any two sides is constant.
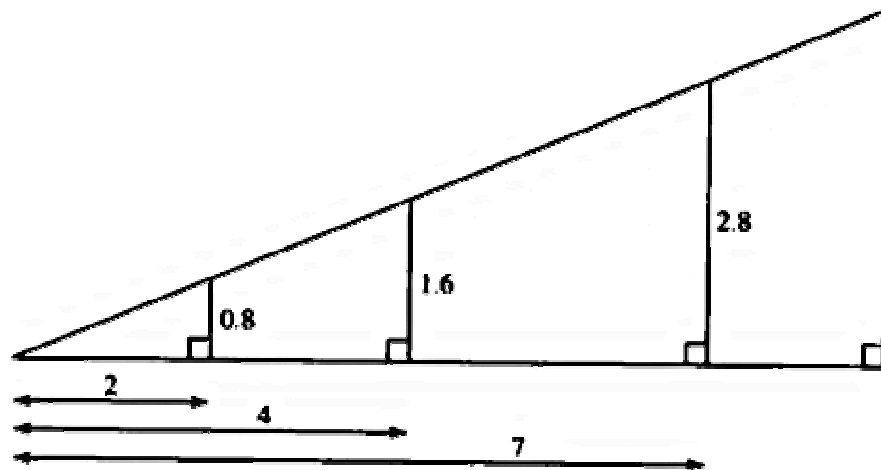
Figure 6.3 Nested similar right-angled triangles. The ratio of any two sides in each of the triangles is the same. The symbol □ indicates a right angle. Notice that the ratio of the height to the base in each triangle is the same. $\frac{0.8}{2} = \frac{1.6}{4} = \frac{2.8}{7} = 0.4$

Example 6.1 A right-angled triangle has an internal angle of 30°. Find the ratio of the side opposite this angle to the hypotenuse.

SOLUTION. First, draw any triangle with a 30° angle and a right angle (90°) as in Fig. 6.4. The hypotenuse is the side opposite the right angle. The triangle in Fig. 6.4 has a side opposite the $30°$ angle of length 1.75 cm and a hypotenuse of length 3.5 cm. The ratio is therefore $1.75/3.5 = 0.5$
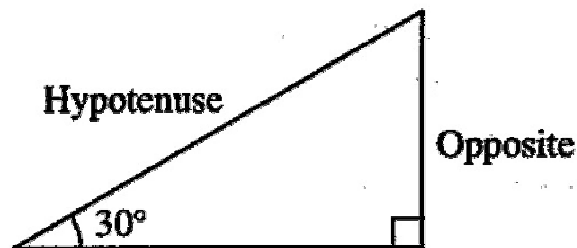
Figure 6.4 A right-angled triangle with a 30^0 angle.

6.4 THE RATIOS OF THE SIDES

Each of the ratios of a right-angled triangle has a name. The most important to remember are the sine, cosine and tangent; the others, cosecant, secant and cotangent are the reciprocals of these. The names are abbreviated to sin, cos, tan, cosec, sec and cotan. To see how these are all defined refer to Fig. 6.5.

$$\sin(\alpha) = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos(\alpha) = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan(\alpha) = \frac{\text{opposite}}{\text{adjacent}}$$

$$\operatorname{cosec}(\alpha) = \frac{\text{hypotenuse}}{\text{opposite}} \qquad \sec(\alpha) = \frac{\text{hypotenuse}}{\text{adjacent}} \qquad \operatorname{cotan}(\alpha) = \frac{\text{adjacent}}{\text{opposite}}$$
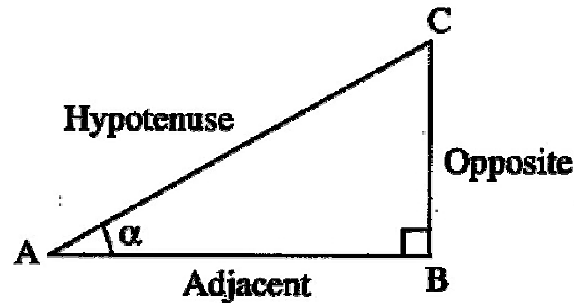
Figure 6.5 A right-angled triangle and the names given to the ratios of the sides (trigonometric ratios).

The values of these ratios can be found by using a calculator and are used to find the length of a side of the triangle when one side and an angle are known or to find the angle when two of the sides are known.

Example 6.2 Find the length of the side indicated in the triangle in Fig. 6.6
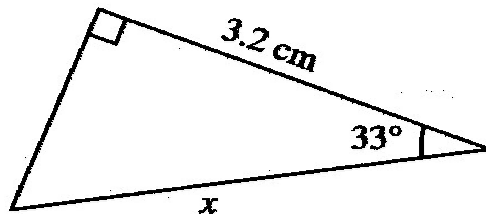
Figure 6.6 The triangle for Example 6.2

SOLUTION The unknown side, x, is the hypotenuse. The marked side of 3.2cm is adjacent to the given angle of 33°. One trigonometric ratio containing the adjacent and hypotenuse is the cosine.

$$\cos 33^\circ = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{3.2}{x}$$

$$\Rightarrow 0.8387 = \frac{3.2}{x}$$

$$\Leftrightarrow x = \frac{3.2}{0.8387} \approx 3.82 \text{ to two decimal places}$$

So the length of the side is 3.82 to two decimal places.

Example 6.3 Find the length of the side indicated in the triangle in Fig. 6.7
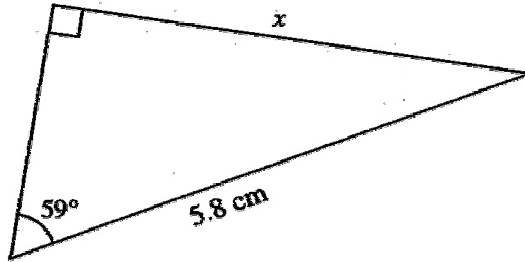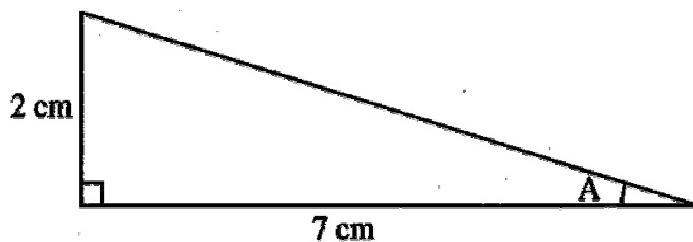

Figure 6.7 The triangle used for Example 6.3

SOLUTION The unknown side, x, is opposite to the given angle of 59°. The hypotenuse is given as 5.8cm. One trigonometric ratio containing the opposite and hypotenuse is the sine, giving

$$\sin 59^\circ = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{x}{5.8}$$

$$0.8572 = \frac{x}{5.8} \Rightarrow x = 4.97 \text{ to two decimal places}$$

Therefore the length of the side is 4.97cm.

Example 6.4 Find the angle A in the triangle in Fig. 6.8

Figure 6.8 **The triangle for Example 6.4**

SOLUTION The side marked as 2cm is opposite the unknown angle A. The side marked as 7cm is adjacent to the unknown angle A. One trigonometric ratio containing the opposite and adjacent is the tangent:

$$\tan(A) = \frac{\text{opposite}}{\text{adjacent}} = \frac{2}{7} = 0.2857$$

Use the inverse tangent function button to get A = 16°; hence the unknown angle A is 16°.

We can use the ratios of the sides to solve problems where an angle and the length of one side is known, or where two sides are known and we want to find the value of an angle. However, we need some other way to find the third side when two of the sides are known.

6.4 PYTHAGORAS'S THEOREM

Pythagoras's theorem states that the square on the hypotenuse is equal to the sum of the squares on the other two sides. This is expressed as $c^2 = a^2 + b^2$, and the sides are labelled in Fig. 6.9

Pythagoras' theorem is easy to justify and is done so by consideration of the area of a square of side $a + b$ as shown in Fig. 6.10. The area of the total square is given by the product of the sides $(a + b)^2$. Inside the square there are the four triangles of total area

$$4 \times \tfrac{1}{2}(a \times b) = 2ab$$

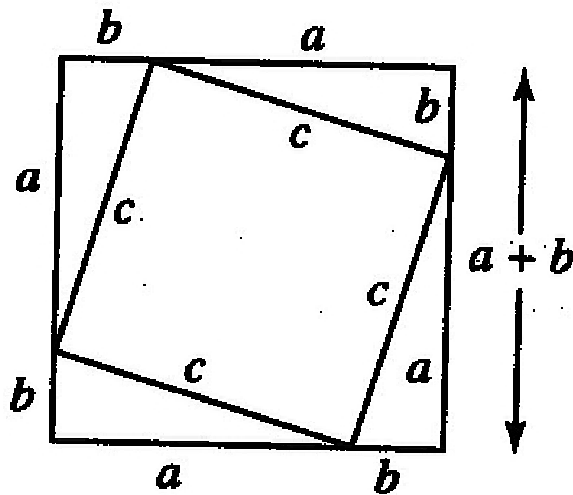(the area of a right-angled triangle is half base × height). The area of the square in the middle is c^2.
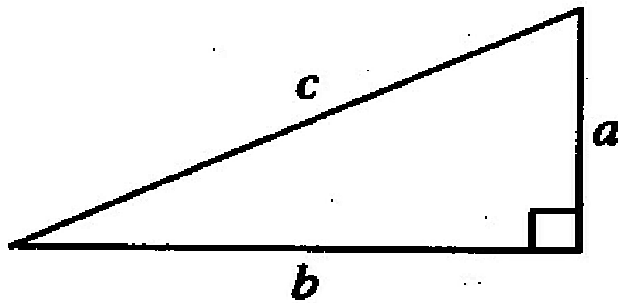
Figure 6.9 Pythagoras' theorem gives that $c^2 = a^2 + b^2$

Figure 6.10 To justify Pythagoras' theorem, draw a square of side *a+b* and place triangles of sides *a,b,* and *c* in each corner.

As the total area of the four triangles and the square of side *c* must equal the area of the big square of side *a+b* we have
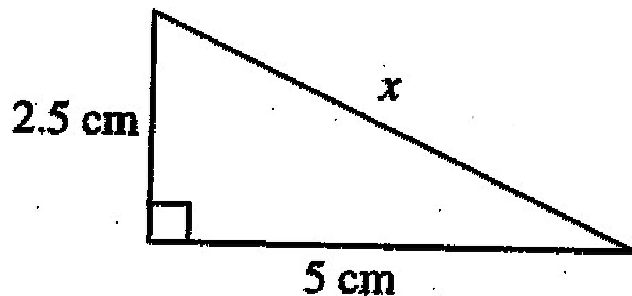
$$(a+b)^2 = 2ab + c^2$$

$$\Leftrightarrow a^2 + 2ab + b^2 = 2ab + c^2$$

Subtracting $2ab$ from both sides gives $a^2 + b^2 = c^2$, which is Pythagoras's theorem.

Pythagoras's theorem can now be used to find the third side given the length of the other two.

Example 6.5 Find the length of the side indicated in Fig. 6.11

Figure 6.11 The triangle used for Example 6.5

SOLUTION x is the hypotenuse. It therefore takes the place of c in Pythagoras's formula:

$$a^2 + b^2 = c^2 \quad \text{and } a = 2.5, b = 5$$

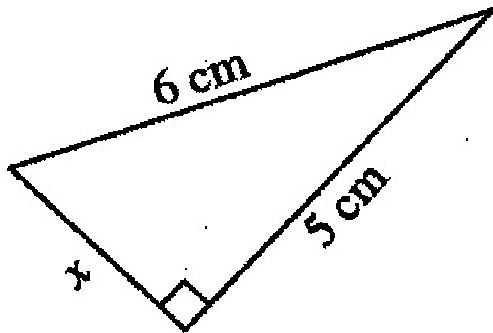$$\Leftrightarrow (2.5)^2 + 5^2 = x^2$$

$$\Leftrightarrow x^2 = 6.25 + 25$$

$$= 31.25$$

$$\Rightarrow x = \sqrt{31.25} \approx 5.6 \qquad \text{(where, as } x \text{ represents a length, } x \geq 0)$$

The length of the hypotenuse is 5.6cm to two significant figures.

Example 6.6 Find the length of the side indicated in Fig. 6.12

Figure 6.12 The triangle for Example 6.6

SOLUTION x is not the hypotenuse, and therefore takes the place of a or b in Pythagoras's formula, while 6 takes the place of c, giving

$$x^2 + 5^2 = 6^2$$

$$\Leftrightarrow x^2 + 25 = 36$$

$$\Leftrightarrow x^2 = 11$$

$$\Rightarrow x = \sqrt{11} \approx 3.32 \qquad \text{(where } x \geq 0)$$

The length of the side is 3.32 cm to two decimal places.

Example 6.7 A ladder of length 6 m is placed against a wall. The greatest angle it can make to the ground and still be stable is 70°.

(a) Find the maximum reach of the ladder vertically up the wall.

(b) What length of ladder would be needed to reach a window at a height of 20m when the foot of the ladder cannot be placed any nearer than 2 m away from the wall?

These problems can be pictured as in Fig. 6.13.
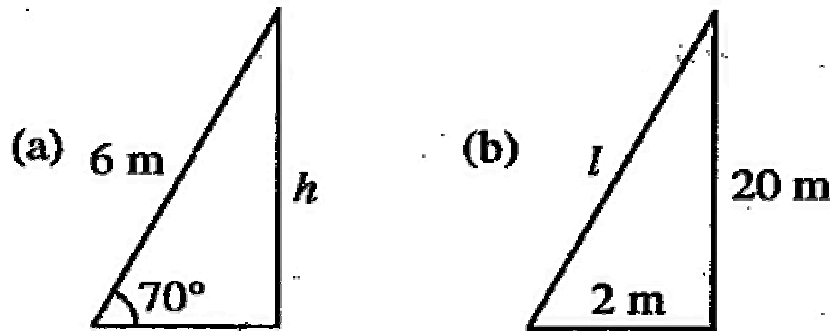
Figure 6.13 (a) The maximum angle of the ladder is 70° (b) A ladder is needed to reach a window at a height of 20 m.

SOLUTION

(a) The unknown height is marked as h in Fig. 6.13(a). h is opposite the 70° angle and the ladder of length 6 m forms the hypotenuse. Using the sine we get

$$\sin(70°) = \frac{h}{6} \Leftrightarrow h = 6\sin(70°) \Rightarrow h \approx 5.64 \text{ to two d.p.}$$

The height reached by the ladder is 5.64 m to two decimal places.

(b) The length of the ladder, marked as l, forms the hypotenuse in Fig. 6.13(b). As we know the lengths of the other two sides we use Pythagoras's theorem giving

$$l^2 = 2^2 + 20^2$$

$$\Leftrightarrow l^2 = 4 + 400 \Leftrightarrow l^2 = 404$$

$$\Rightarrow l = 20.1 \text{ to two d.p.}$$

The length of the ladder is 20.1 m to two decimal places.

Example 6.8 In the triangle shown in Fig. 6.14, find the length of the side marked c.
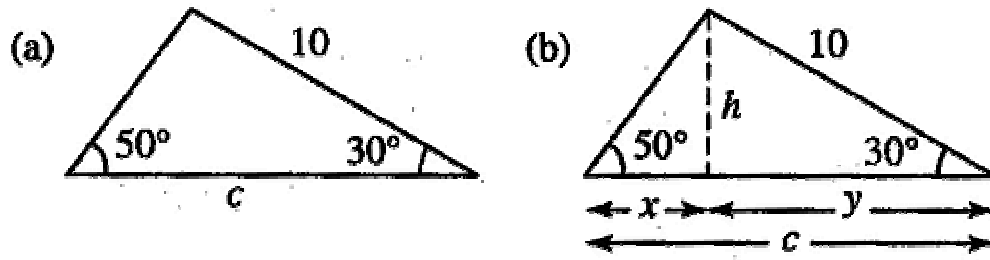
Figure 6.14 (a) The triangle for Example 6.8. We wish to find the length marked *c*. (b)The same triangle with a perpendicular drawn and unknown lengths, *h*, *x* and *y* marked. Notice that *c* = *x* + *y*.

SOLUTION Split the triangle into two right-angled triangles by dropping a perpendicular as in Fig. 6.14(b). The length, *c*, is given by, $c = x + y$ however *h* needs also to be calculated in order to find *x*.

From the right angled triangle with the 30° angle:

$$\sin(30°) = \frac{h}{10} \Leftrightarrow h = 10\sin(30°) \Leftrightarrow h = 5$$

Also from Pythagoras

$$h^2 + y^2 = 10^2$$

Substituting for *h* gives

$$5^2 + y^2 = 10^2 \Leftrightarrow y^2 = 100 - 25$$

$$\Leftrightarrow y^2 = 75 \Rightarrow y \approx 8.66 \text{ to two d.p.}$$

From the triangle with the 50° angle:

$$\tan(50°) = \frac{h}{x} \Leftrightarrow x = \frac{5}{\tan(50°)} \Rightarrow x \approx 4.2$$

We are now able to find the length of the side *c*, using $c = x + y$ giving $c = 8.66 + 4.2 = 12.86$. The length of the side marked *c* in the triangle is 12.86

6.6 SUMMARY

1. Right angled triangles can be solved if one angle (other than the right angle) is known and one side, or if two sides are known.
2. The trigonometric ratios are used to solve problems where one angle is known and one side and we wish to find the length of one of the other sides. They are also used to find an angle when the length of two sides is known.

3. The most important trigonometric ratios are:-

$$\sin(\alpha) = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos(\alpha) = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan(\alpha) = \frac{\text{opposite}}{\text{adjacent}}$$

and the others are given in the caption to Figure 6.5

4. Pythagoras' theorem is used to find a third side in a right-angled triangle when the other two are known. It is expressed by

$$a^2 + b^2 = c^2$$

where c is the hypotenuse.

5. Many other geometrical problems can be solved by splitting shapes into right-angled triangles.

6.7 EXERCISES

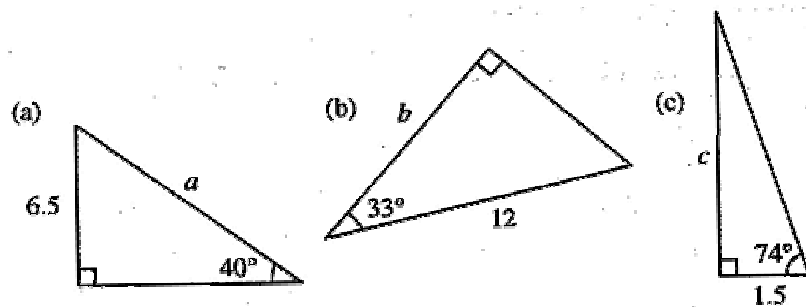6 1.Find the sides of the triangles shown in Figure 6.15 (a), (b) and (c)

Figure 6.15 Triangles for Exercise 6.1

6.2. Find the angles in the triangles shown in Figure 6.16
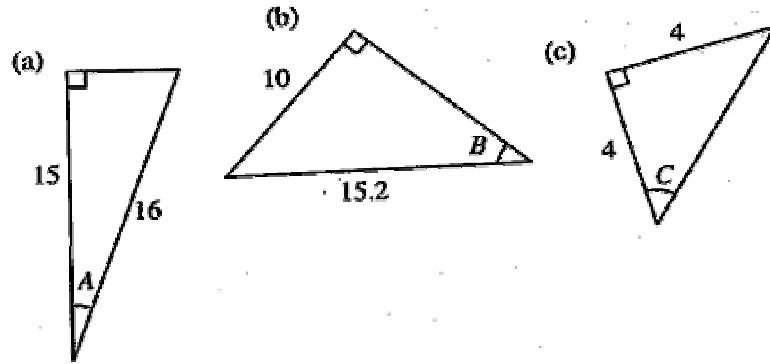
Figure 6.16 Triangles for Exercise 6.2

6.3. Find the unknown side in the triangles shown in Figure 6.17(a), (b) and (c).
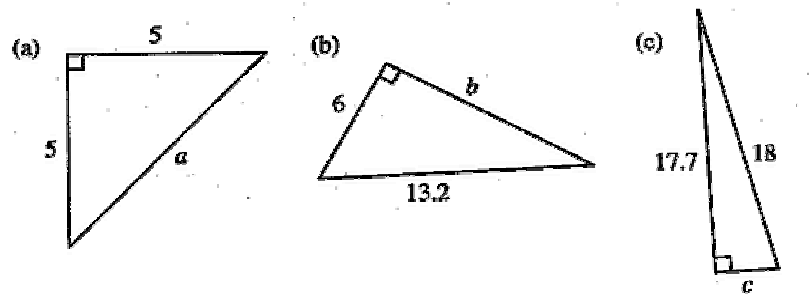

Figure 6.17 Triangles for Exercise 6.3

6.4. A road has a gradient of 1 in 10 (that is it rises 1 m for every 10 m horizontally). If a car travels 30m along the road, how far has it travelled in a horizontal direction and how far has it risen?

6.5. A statue is such that at a certain time of day, when the sun makes an angle of approximately $63°$ with the ground, the length of its shadow is 10.2m. Estimate the height of the statue.

6.6. Two forces F_1 , F_2 are at right angles and the magnitude of the resultant force is given by the length of the diagonal as shown in Figure 6.18. If $F_1 = 3.6$ N and $F_2 = 5.4$ N find the magnitude of F_R
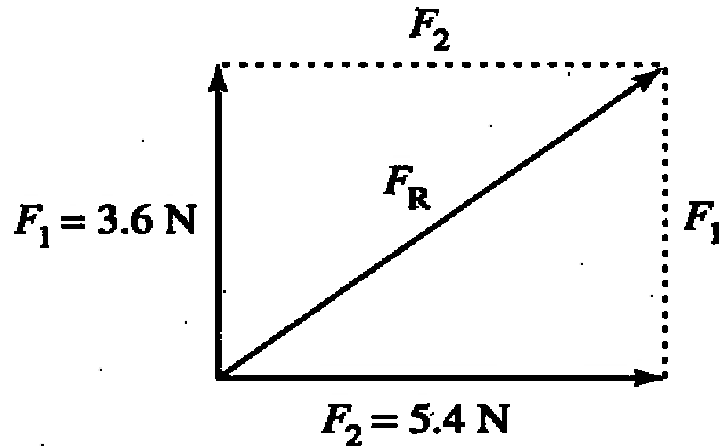
Background Mathematics Notes for Mathematics for Electrical Engineering and Computing by Mary Attenborough

Figure 6.18 Forces for Exercise 6.6. F_R is the resultant of forces F_1 and F_2 at right angles.

6.7. A generator of frequency 20Hz is applied to a series circuit as shown in Figure 6.19(a). The resistor has a resistance of 5.3Ω and at this frequency the inductor is found to have a reactance of 294.1Ω. The resultant voltage V can be found by solving the triangle as shown in Figure 6.19(b) where V_L is the voltage across the inductor ($V_L = 294.1\,I$) and V_R is the voltage across the resistor ($V_R = 5.3I$) and I is the current in the circuit. Find the size of V in terms of I and also find the phase,ϕ, which gives the angle by which the resultant voltage leads the current.
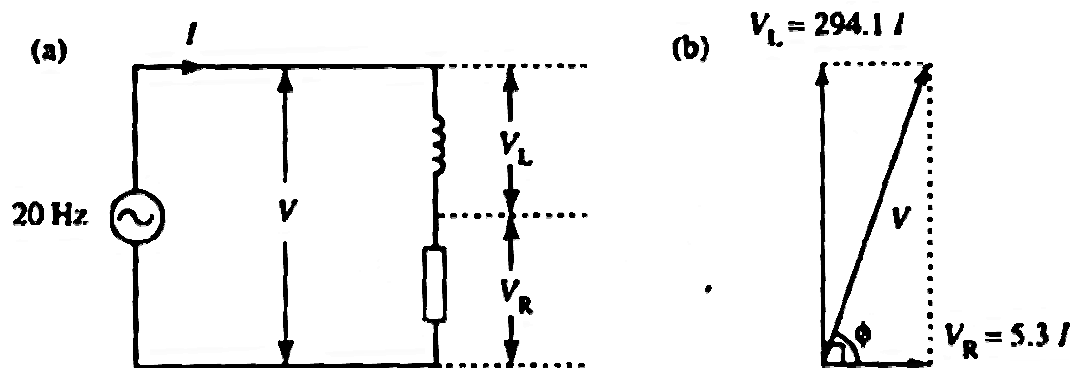

Figure 6.19 (a) Circuit for Exercise 6.7 and b) Triangle of voltages with resultant V

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